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A SOLUTION TO EINSTEIN'S GRAVITATIONAL FIELD EQUATION  
FOR A SPHERICALLY SYMMETRICAL STATIC PERFECT FLUID

---

A Dissertation

Presented to  
the Faculty of the Graduate School  
University of Missouri

---

In Partial Fullfillment  
of the Requirements for the Degree  
Doctor of Philosophy

---

by  
Lionel Donnell Hewett, 1938

July 1965

Jack L. Rivers, Dissertation Supervisor

134473

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GRADUATE SCHOOL

Graduate Form Ph.D. III

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A SOLUTION TO EINSTEIN'S GRAVITATIONAL FIELD EQUATION

FOR A SPHERICALLY SYMMETRICAL STATIC PERFECT FLUID

presented by LIONEL DONNELL HEWETT

a candidate for the degree of Doctor of Philosophy in

PHYSICS

and hereby certify that in their opinion it is worthy of acceptance.

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## ABSTRACT

Given spherical symmetry, Einstein's gravitational field equations are reduced to a single, second order, linear, homogeneous differential equation. The solution of this equation for a homogeneous sphere is identical with Schwarzschild's solution. The equation is solved for a centrally dense star:  $T_4^4 = K(r_0/r)^{-4}$ , with parameters  $K$  and  $r_0$  adjusted to fit the sun. Values of relative density, interior mass, relative pressure, relative and absolute temperature are obtained as functions of radius,  $r$ , and are compared with corresponding values for the homogeneous sphere, and for a model sun.

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## CHAPTER I

### INTRODUCTION

In the year 1916, in Annalen Der Physik, volume 49, Dr. Albert Einstein published a paper entitled "Die Grundlagen der Allgemeinen Relativitatstheorie" which contained the basic concepts, hypotheses, and mathematical framework of a theory relating space, time, and gravitation. The fundamental assumption of this theory, now called "Einstein's General Theory of Relativity", asserts that there exists a unique relationship between the geometrical properties of our four-dimensional space-time and the distribution of mass-energy therein.

The geometrical properties of any space are contained in a symmetrical second rank tensor, called the metric tensor, the covariant components of which,  $g_{ij}$ , are defined through the metric equation:

$$ds^2 = g_{ij} dx^i dx^j, \quad 1.1$$

where  $ds$  is the invariant interval between two infinitesimally separated events, the difference in the  $i$ 'th coordinates of which are  $dx^i$ . (An index occurring both as superscript and subscript within the same term of an expression implies a summation over the repeated index.) The contravariant components of the metric tensor  $g^{ij}$  are defined through the equation:

$$\delta_j^i = g^{ik} g_{kj}, \quad 1.2$$

where  $\delta_j^i$  is the Kronecker delta.

The metric tensor may be used to generate other tensors, many of which contain the same information about the geometrical properties of the space; i.e. one can determine the metric tensor from such derived tensors. One such tensor, called the Ricci tensor, another symmetrical second rank tensor, is related to the metric tensor through the equation:

$$R_{ij} = \frac{\partial}{\partial x^i} \left\{ \begin{matrix} k \\ j \ k \end{matrix} \right\} - \frac{\partial}{\partial x^k} \left\{ \begin{matrix} k \\ i \ j \end{matrix} \right\} + \left\{ \begin{matrix} m \\ i \ k \end{matrix} \right\} \left\{ \begin{matrix} k \\ j \ m \end{matrix} \right\} - \left\{ \begin{matrix} m \\ i \ j \end{matrix} \right\} \left\{ \begin{matrix} k \\ m \ k \end{matrix} \right\}, \quad 1.3$$

where the Christoffel symbol of the second kind  $\left\{ \begin{matrix} k \\ i \ j \end{matrix} \right\}$  is given by

$$\left\{ \begin{matrix} k \\ i \ j \end{matrix} \right\} = \left\{ \begin{matrix} k \\ j \ i \end{matrix} \right\} = \frac{1}{2} g^{km} \left[ \frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right]. \quad 1.4$$

The tensor that describes the distribution of energy in a four-dimensional space is called the stress-momentum-energy density tensor or simply the energy tensor, and at the origin of a local Cartesian coordinate system<sup>1</sup> is given by

$$T^{ij} = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} & P_x \\ S_{yx} & S_{yy} & S_{yz} & P_y \\ S_{zx} & S_{zy} & S_{zz} & P_z \\ P_x & P_y & P_z & \rho \end{pmatrix}, \quad 1.5$$

where the three dimensional second rank tensor

$$S_{ij} = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix} \quad 1.6$$

is the usual stress tensor written in a Cartesian coordinate system, the three dimensional vector

$$P_i = (P_x, P_y, P_z) \quad 1.7$$

is the momentum density vector, and the scalar  $\rho$  is the mass density.

Einstein's fundamental assumption in general relativity can now be expressed mathematically as an equation relating the Ricci tensor  $R_{ij}$  with the energy tensor  $T^{ij}$  as follows:

$$- 8\pi G T_{ij} = R_{ij} - \frac{1}{2} g_{ij} R^k_k, \quad 1.8$$

where  $G$ , the universal gravitational constant, equals  $6.670 \times 10^{-11} \text{ n-m}^2/\text{kg}^2$ . This equation, called Einstein's gravitational field equation, is seen to be a second order non-linear differential equation relating the two tensors: the energy tensor and the metric tensor.

For a given metric tensor it is a straightforward but tedious matter to determine the energy tensor associated with the space. Such energy tensors, however, are in general found to give complex, non-physical stresses in the space. The types of materials that would go into the make up of such spaces are seldom found in nature. For this reason, it is frequently desired to know the geometrical properties of a space of a given mass distribution.

In 1916 Schwarzschild solved Einstein's gravitational field equation in the region outside a spherically symmetrical distribution of matter.<sup>2</sup> (Later that year Droste obtained the same solution independently.<sup>3</sup>) The solution contains one arbitrary constant which is related

to the Newtonian gravitational mass of the system. The solution also displays a singular region about the center of symmetry, thereby implying a minimum diameter for a spherical body of a given mass.

Later that same year Schwarzschild obtained a second solution to Einstein's equation.<sup>4</sup> This time he considered the case of the homogeneous sphere of constant density. This solution also displays a singular region, which determines a maximum diameter for a body of a given density.

By combining the two solutions, Schwarzschild was able to obtain the first solution to Einstein's gravitational field equation for a given mass distribution, valid for an entire space.

The methods used by Schwarzschild to solve Einstein's equation were peculiar to the two types of mass distributions and were not extendable to other, more general mass distributions. Apparently, little progress has been made toward other analytic solutions of Einstein's equations since that time. Most of the more recent attempts to solve Einstein's equations have dealt with methods of approximation, and most of these approximations consist of assuming low mass density, or small total masses, in which case the curved space-time is approximately Minkowskian.

There is, therefore, a need for a straightforward method of solving Einstein's gravitational field equation. This paper is concerned with one such method which is applicable to a large variety of mass distributions, such as slowly rotating stars, globular clusters, spherical

galaxies, and possibly the universe as a whole. In particular, for the case of a spherically symmetrical static perfect fluid Einstein's equations, which consist of the simultaneous solution of several second order non-linear differential equations, have been reduced to the solution of a single second order linear equation. For many cases a power series solution is applicable and relatively straightforward.

Knowing the components of the metric tensor, one may then find other properties of the system such as: (1) the mass or energy density, (2) the pressure of the fluid constituting the system, (3) the temperature of the fluid (assuming an equation of state), (4) the limiting size of a spherically symmetrical body of a particular type.

## CHAPTER II

### EINSTEIN'S GRAVITATIONAL FIELD EQUATION IN SCHWARZSCHILD COORDINATES

#### A. Spherically Symmetric Metric Tensor

Through the utilization of symmetry properties it has been shown<sup>5</sup> that the most general form of the metric for a spherically symmetrical static space-time may be written in a set of orthogonal coordinates as:

$$ds^2 = e^h dt^2 - \frac{1}{c^2} e^f dr^2 - \frac{1}{c^2} e^g (r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2), \quad 2.1$$

where  $f, g, h$  are arbitrary functions of  $r$ , and  $c$ , the velocity of light, equals  $2.997930 \times 10^8$  m/sec. Such a coordinate system has its spacial origin at the center of symmetry. One sees that  $t$  is the time-like coordinate and  $r, \theta, \phi$  are the three space-like coordinates. Furthermore, in the weak field case, where the space-time becomes Minkowskian ( $f = g = h = 0$ ), it is evident that  $r, \theta, \phi$  approach the usual spherical polar coordinates. We shall, therefore, refer to  $r$  as the radial coordinate even though it cannot, in general, be interpreted as the "distance from the origin", to  $\theta$  as the polar angle, and to  $\phi$  as the azimuthal angle.

By inspection the covariant and contravariant components of the metric tensor may be written in the following matrix arrays:

$$g_{ij} = \begin{pmatrix} -\frac{1}{c^2}e^f & 0 & 0 & 0 \\ 0 & -\frac{1}{c^2}e^{gr^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{c^2}e^{gr^2}\sin^2\theta & 0 \\ 0 & 0 & 0 & e^h \end{pmatrix}, \quad 2.2$$

$$g^{ij} = \begin{pmatrix} -c^2e^{-f} & 0 & 0 & 0 \\ 0 & -c^2e^{-gr^2} & 0 & 0 \\ 0 & 0 & -c^2e^{-gr^2}\sin^{-2}\theta & 0 \\ 0 & 0 & 0 & e^{-h} \end{pmatrix}. \quad 2.3$$

The non-vanishing components of the Ricci tensor are found to be

$$\begin{aligned} R_{11} &= g'' + \frac{1}{2}h'' + \frac{2}{r}g' - \frac{1}{r}f' - \frac{1}{2}f'g' - \frac{1}{4}f'h' + \frac{1}{4}h'^2 + \frac{1}{2}g'^2 \\ R_{22} &= e^{g-f} \left[ 1 + 2rg' + \frac{1}{2}r(h'-f') + \frac{1}{2}r^2g'' + \frac{1}{2}r^2g'(g' + \frac{1}{2}h' - \frac{1}{2}f') \right] - 1 \\ R_{33} &= R_{22} \sin^2\theta \\ R_{44} &= -c^2e^{h-f} \left[ \frac{1}{2}h'' + \frac{1}{r}h' + \frac{1}{2}h'g' - \frac{1}{4}f'h' + \frac{1}{4}h'^2 \right], \end{aligned} \quad 2.4$$

where the primes denote differentiation with respect to  $r$ .

Because the vectorial divergence of the energy tensor vanishes identically,<sup>6</sup> Einstein's gravitational field equations (assuming the  $T_{ij}$  are known functions of  $r$ ) give only two independent equations for the determination of the three unknown functions  $f$ ,  $g$ ,  $h$ . One is, therefore, at liberty to choose arbitrarily a third equation in order to determine



uniquely these three functions. In general, there are four arbitrary relations corresponding to the arbitrary choice of four coordinates.

In this case, however, the metric is already determined as far as  $t$ ,  $\theta$ ,  $\phi$  are concerned. The choice of the third relation determines which of the possible choices for a radial coordinate we are going to use.

### B. Isotropic Coordinates

Isotropic coordinates are defined when one chooses the third equation to be  $f = g$  and, in order to prevent confusion with other coordinate systems, places a bar above each of the variables. In isotropic coordinates the metric equation becomes:

$$ds^2 = \bar{h} \bar{dt}^2 - \frac{1}{c^2} \bar{e} \bar{f} (\bar{dr}^2 + \bar{r}^2 \bar{d\theta}^2 + \bar{r}^2 \sin^2 \bar{\theta} \bar{d\phi}^2) . \quad 2.5$$

This coordinate system has the intuitively desirable property that the coordinate length of a small "rigid rod" ( $ds = \text{const.}$ ) does not alter when the orientation of the rod is altered; the speed of light is independent of direction. One sees, therefore, that the isotropic coordinate system is the natural or intuitive polar coordinate system, so that one would, if possible, like to solve his problems in this system of coordinates.

### C. Schwarzschild Coordinates

Another coordinate system, first utilized by Schwarzschild in his solution to Einstein's equation for a vacuum, is determined by setting  $g = 0$ . In this coordinate system the metric equation becomes:

$$ds^2 = e^h dt^2 - \frac{1}{c^2} (e^f dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) . \quad 2.6$$

Since  $g = 0$ , this coordinate system frequently simplifies the equations and reduces the mathematical travail necessary to solve Einstein's equation. For this reason, we will find it expedient to utilize Schwarzschild coordinates when solving Einstein's equation. One may, of course, whenever he wishes, transform to either isotropic coordinates or local Cartesian coordinates.

To transform between Schwarzschild coordinates and isotropic coordinates, one may use the transformation:

$$\begin{aligned} r &= \bar{r} \exp \frac{1}{2} \bar{f} & \bar{r} &= \exp \int_{+\infty}^r \frac{1}{r} \exp \frac{f}{2} dr \\ \theta &= \bar{\theta} & \bar{\theta} &= \theta \\ \phi &= \bar{\phi} & \bar{\phi} &= \phi \\ t &= \bar{t} & \bar{t} &= t . \end{aligned} \quad 2.7$$

To transform between Schwarzschild coordinates and local Cartesian coordinates, one uses the following transformation:<sup>1</sup>

$$\begin{aligned} r &= x + r_o & x &= r - r_o \\ \theta &= \frac{y}{r_o} + \theta_o & y &= r_o (\theta - \theta_o) \\ \phi &= \frac{z}{r_o \sin \theta_o} & z &= r_o \sin \theta_o (\phi - \phi_o) . \end{aligned} \quad 2.8$$

The point  $r_0, \theta_0, \phi_0$  is, of course, the origin of the local Cartesian coordinate system. At this point the metric equation becomes:

$$ds^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) . \quad 2.9$$

#### D. Transformation of Mixed Diagonal Second Rank Tensor

Since the components of the energy tensor (as well as almost all tensors) are highly dependent upon the coordinates chosen, in order to solve Einstein's equation, one must first know the components of the energy tensor in the coordinate system being used. Because of the intuitive nature of the isotropic coordinate system, one might suppose that he must assume an energy tensor in isotropic coordinates, transform to a more tractable coordinate system such as Schwarzschild coordinates, solve Einstein's equation for the components of the metric tensor, and then transform back to the isotropic coordinate system (or to a local Cartesian coordinate system) in order to interpret the results. However, this is not at all necessary, for we shall see presently that by using the mixed form of Einstein's equation, the components of the energy tensor are the same in both isotropic coordinates and Schwarzschild coordinates, and we shall see later that a storehouse of information can be extracted directly from the Schwarzschild coordinate solution without the necessity of transforming first to either isotropic or local Cartesian coordinates. In other words, one is able to use Schwarzschild coordinates

exclusively.

Rather than restrict ourselves to proving that the values of the corresponding components of our energy tensor are the same in Schwarzschild coordinates as in isotropic coordinates, let us consider the more general case. Let  $x^i$  and  $\bar{x}^i$  be two coordinate systems related through the transformation equations

$$\begin{aligned}\bar{x}^1 &= \bar{x}^1(x^1) & x^1 &= x^1(\bar{x}^1) \\ \bar{x}^2 &= \bar{x}^2(x^2) & x^2 &= x^2(\bar{x}^2) \\ \bar{x}^3 &= \bar{x}^3(x^3) & x^3 &= x^3(\bar{x}^3) \\ \bar{x}^4 &= \bar{x}^4(x^4) & x^4 &= x^4(\bar{x}^4)\end{aligned}\tag{2.10}$$

so that each  $\bar{x}^i$  is a function solely of the corresponding  $x^i$  and vice versa. Further let  $T_j^i$  and  $\bar{T}_j^i$  be the mixed components of a second rank tensor in the respective coordinate systems, and assume  $T_j^i$  is diagonal:

$$T_j^i = T_i^i \delta_j^i \quad (\text{no sum on } i) .$$

Then according to the rules for transforming components of tensors, the components  $\bar{T}_j^i$  are

$$\begin{aligned}\bar{T}_j^i &= T_m^k \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^m}{\partial \bar{x}^j} \\ &= T_j^i \frac{dx^i}{dx^i} \frac{dx^j}{dx^j} \quad (\text{no sum on } i, j) \\ &= T_i^i \frac{dx^i}{dx^i} \frac{dx^i}{dx^i} \delta_j^i \quad (\text{no sum on } i) \\ \bar{T}_j^i &= T_j^i .\end{aligned}\tag{2.11}$$

Therefore, the value of the mixed components of a second rank tensor is independent of the coordinate system as long as each of the coordinates of the first coordinate system is a function solely of the respective coordinate in the second coordinate system and the tensor is diagonal in one of the coordinate systems.

Obviously the Schwarzschild coordinate system and the isotropic coordinate system are so related, and we shall see that in Schwarzschild coordinates the energy tensor is diagonal, so that as far as the energy tensor is concerned there is no practical advantage in using isotropic coordinates over Schwarzschild coordinates.

#### E. The Energy Tensor

Let us now proceed to obtain Einstein's gravitational field equations in Schwarzschild coordinates. From equations 2.2 and 2.3, one finds the metric tensor to have the components

$$g_{ij} = \begin{pmatrix} -\frac{1}{c^2}e^f & 0 & 0 & 0 \\ 0 & -\frac{1}{c^2}r^2 & 0 & 0 \\ 0 & 0 & -\frac{1}{c^2}r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & e^h \end{pmatrix}, \quad 2.12a$$

and

$$g^{ij} = \begin{pmatrix} -c^2 e^{-f} & 0 & 0 & 0 \\ 0 & -c^2 r^{-2} & 0 & 0 \\ 0 & 0 & -c^2 r^{-2} \sin^2 \theta & 0 \\ 0 & 0 & 0 & e^{-h} \end{pmatrix}, \quad 2.12b$$

and the non-vanishing components of the Ricci tensor to be given by

$$\begin{aligned} R_{11} &= \frac{1}{2} h'' - \frac{1}{r} f' - \frac{1}{4} f' h' + \frac{1}{4} h'^2 \\ R_{22} &= e^{-f} + \frac{1}{2} r e^{-f} (h' - f') - 1 \\ R_{33} &= R_{22} \sin^2 \theta \\ R_{44} &= -c^2 e^{h-f} \left( \frac{1}{2} h'' + \frac{1}{r} h' - \frac{1}{4} f' h' + \frac{1}{4} h'^2 \right). \end{aligned} \quad 2.13$$

Let us now make the following definitions:

$$\begin{aligned} e^f &= \frac{1}{J} \\ e^h &= J e^b \\ J &= 1 - I \\ I &= \frac{a}{r}, \end{aligned} \quad 2.14$$

where  $a$  and  $b$  are arbitrary functions of  $r$ . In terms of our new functions, the non-vanishing covariant components of the Ricci tensor are:

$$\begin{aligned} R_{11} &= -\frac{a''}{2rJ} + \frac{1}{2} b'' + \frac{1}{4} b'^2 - \frac{3}{4} b' f' \\ R_{22} &= -a' + \frac{1}{2} r J b' \\ R_{33} &= R_{22} \sin^2 \theta \\ R_{44} &= -c^2 J^2 e^b (R_{11} + \frac{b'}{r}). \end{aligned} \quad 2.15$$

The mixed components of the Ricci tensor, defined as

$$R_j^i = g^{ik} R_{kj} ,$$

are then given by

$$\begin{aligned} R_1^1 &= c^2 \left( \frac{a''}{2r} - \frac{Jb''}{2} - \frac{Jb'^2}{4} + \frac{3a'b'}{4r} - \frac{3ab'}{4r^2} \right) \\ R_2^2 &= c^2 \left( \frac{a'}{r^2} - \frac{Jb'}{2r} \right) \\ R_3^3 &= R_2^2 \\ R_4^4 &= R_1^1 - c^2 \frac{Jb'}{r} \\ R_j^i &= 0 \quad \text{for } i \neq j. \end{aligned} \tag{2.16}$$

The trace of the mixed form of the Ricci tensor, called the curvature invariant, is

$$R_k^k = 2(R_1^1 + R_2^2) - c^2 J \frac{b'}{r} . \tag{2.17}$$

The mixed form of Einstein's gravitational field equations,

$$- 8\pi G T_j^i = R_j^i - \frac{1}{2} \delta_j^i R_k^k , \tag{2.18}$$

is now found to be

$$\begin{aligned} - 8\pi G T_1^1 &= - c^2 \left( \frac{a'}{r^2} - J \frac{b'}{r} \right) \\ - 8\pi G T_2^2 &= - c^2 \left( \frac{a''}{2r} - \frac{Jb'}{2} - \frac{Jb'^2}{4} + \frac{3a'b'}{4r} - \frac{b'}{2r} - \frac{ab'}{4r^2} \right) \\ - 8\pi G T_3^3 &= - 8\pi G T_2^2 \\ - 8\pi G T_4^4 &= - c^2 \frac{a'}{r^2} \\ - 8\pi G T_j^i &= 0 \quad \text{for } i \neq j. \end{aligned} \tag{2.19}$$

The energy tensor is seen to be diagonal in Schwarzschild coordinates, which is required if it is to have the same value as in isotropic coordinates. We note that  $T_2^2 = T_3^3$ , a result not at all surprising considering the spherical symmetry. We note also, that one is not at liberty to choose all the  $T_j^i$  at will in Schwarzschild coordinates. In fact, only two of the three functions  $T_1^1$ ,  $T_2^2$ ,  $T_3^3$  may be chosen arbitrarily, the other being determined through Einstein's equation.



### CHAPTER III

#### SOLUTION OF EINSTEIN'S EQUATION

##### A. Solution for $a(r)$

Assuming that one knows the mass density  $T_4^4$  as a function of the radial coordinate, then he may solve equation 2.19 for the function  $a(r)$ :

$$a(r) = a(0) + \frac{8\pi G}{c^2} \int_0^r T_4^4 r^2 dr, \quad 3.1$$

where  $a(0)$  is the value of  $a$  at  $r = 0$ . Since a space of uniform character requires that

$$g^{11} = -c^2 \left( 1 - \frac{a(r)}{r} \right)$$

remain negative and finite, then

$$a(0) = 0;$$

so that equation 3.1 becomes

$$a(r) = \frac{8\pi G}{c^2} \int_0^r T_4^4 r^2 dr. \quad 3.2$$

Since  $T_4^4$ , the mass density, is never negative,  $a(r)$  is always a positive quantity; furthermore it is seen to continually increase as one encompasses more and more of the material during the integration to larger and larger radii.  $a(r)$ , therefore, is a measure of the amount of mass located within a surface of constant radius  $r$ . In fact, one

notices that if the space were Minkowskian, with volume element  $r^2 \sin\theta \, dr \, d\theta \, d\phi$ , then the total mass enclosed inside a sphere of radius  $r$  would be

$$M(r) = 4\pi \int_0^r T_4^4 r^2 \, dr . \quad 3.3$$

In terms of this "mass", which one may call the gravitational mass enclosed within the surface  $r = \text{const.}$ , the function  $a(r)$  may be written as

$$a(r) = \frac{2G}{c^2} M(r) . \quad 3.4$$

The gravitational mass is found to be that quantity which reduces to the mass occurring in Newton's gravitational law whenever  $r$  has become large enough for Newton's gravitational law to be applicable. It is intriguing that regardless of the complexity of the mass density (given that it be static and spherically symmetrical) it will warp the space-time of general relativity in such a way that the effective gravitational mass is equivalent to the mass obtained by integrating the same mass density (in Schwarzschild coordinates) over a flat, Minkowski space-time.

### B. Perfect Fluid

In order to solve Einstein's equation for the function  $b(r)$ , one must either make further assumptions about the energy tensor (i.e. assume some function for the radial stress  $T_1^1$  or some function for the tangential stress  $T_2^2$ ), or he must make an assumption as to the nature

of the material constituting the spherically symmetrical body. Because most bodies large enough to support a strong gravitational field are either gaseous or at least constructed of material in a state of high fluidity, the most useful assumption would be that the body be composed of a perfect fluid. A perfect fluid, in addition to being unable to maintain a shearing stress or a tensile stress, transmits its compressive stress equally in all directions so that a single, non-negative quantity, called the pressure, completely describes the stress tensor. If  $p$  represents the pressure of the perfect fluid in Schwarzschild coordinates, then one has

$$T_1^1 = T_2^2 = T_3^3 = -\frac{p}{c^2} . \quad 3.5$$

Combining equations 3.5 and 2.19, one readily obtains a single second order non-linear differential equation for the function  $b(r)$  in terms of the already determined function  $a(r)$  and its derivatives:

$$2(r^2 - ar)b'' + (r^2 - ar)b'^2 - (3ra' - 5a + 2r)b' = 2ra'' - 4a'. \quad 3.6$$

By transforming to a new variable

$$y = e^{b/2}$$

so that

$$b = 2 \ln y$$

$$b' = 2 y'/y$$

$$b'' = 2 y''/y - 2 y'^2/y^2 ,$$

one is able to reduce equation 3.6 to a second order linear homogeneous differential equation:

$$2r(r - a)y'' + (5a - 3ra' - 2r)y' + (2a' - ra'')y = 0 , \quad 3.7a$$

$$\text{or} \quad 2r^2 J y'' + (3r^2 J' - 2rJ)y' + [r^2 J'' + 2(1-J)]y = 0 . \quad 3.7b$$

The boundary conditions on the above equations occur at large distances from the origin where the space must become Minkowskian. One would require that

$$g_{11} = -\frac{1}{c^2} J$$

approach  $-1/c^2$  and

$$g_{44} = y^2 J$$

approach 1 as  $r$  approaches infinity. This requires that

$$J = 1 \quad \text{for } r = \infty \quad 3.8a$$

$$y = 1 \quad \text{for } r = \infty . \quad 3.8b$$

If the mass density  $T_4^4$  is sectionally continuous, having no infinite discontinuities, then  $a(r)$  is always positive and continuous. Since a space-time of uniform character requires that  $g_{11}$  and  $g_{44}$  be continuous and non-zero, then both  $J$  and  $y$  are continuous and

$$0 < J \leq 1 \quad \text{for all } r, \quad 3.9a$$

$$\text{and} \quad 0 < y < \infty \quad \text{for all } r. \quad 3.9b$$

One could have chosen the value of  $y$  at  $r$  equal to infinity to be  $-1$ , but since  $y$  can never change sign, it would always be negative. The negative of such a solution is also a solution, so there is no loss of generality in assuming  $y$  to be positive.

Equations 3.8b and 3.9b are the boundary conditions used to evaluate the two arbitrary constants obtained in the solution of equation 3.7.

### C. Transformation of Independent Variable

For specific problems, the boundary condition that  $y$  approach unity as  $r$  approaches infinity may be rather difficult to apply. This is especially true if one seeks a power series solution of equation 3.7. We therefore consider the transformation to a new independent variable defined as

$$x = \frac{r_0}{r_0 + r}, \quad 3.10$$

where  $r_0$  is a constant. One sees that  $x$  ranges from unity to zero as  $r$  ranges from zero to infinity, and the boundary conditions 3.8b and 3.9b become

$$\begin{aligned} y(0) &= 1, \\ 0 &< y(x) < \infty. \end{aligned} \quad 3.11$$

If one lets  $Z$  represent any function of  $r$  and the subscript  $x$  mean the derivative with respect to  $x$ , one finds:

$$\begin{aligned} x' &= -\frac{x^2}{r_0} \\ x'' &= 2\frac{x^3}{r_0^2} \\ r &= r_0(1-x)/x \\ Z' &= -\frac{x^2}{r_0} Z_x \\ Z'' &= \frac{x^4}{r_0^2} Z_{xx} + 2\frac{x^3}{r_0^2} Z_x. \end{aligned} \quad 3.12$$

Substituting 3.12 into 3.2 gives

$$a(x) = - \frac{8\pi G}{c^2} r_o^2 \int_1^x T_4^4 \frac{(1-x)^2}{x^4} dx , \quad 3.13$$

and into 3.7 gives

$$U y_{xx} + V y_x + W y = 0 ,$$

where

$$U = 2x(1-x)J \quad 3.14$$

$$V = 3x(1-x)J_x + 2(3-2x)J$$

$$W = x(1-x)J_{xx} + 2(1-x)J_x + 2 \frac{1-J}{x(1-x)} .$$

Much work has been done on the solution of second order linear homogeneous differential equations and many powerful methods have been devised for solving such equations. For particular mass distributions some methods will be more applicable than others so that a general solution can be carried no farther than equation 3.7 or the equivalent 3.14.

#### D. General Procedure

To solve Einstein's gravitational field equation it is necessary to know the energy tensor in the coordinate system being used, say Schwarzschild coordinates. In principle one could experimentally measure the  $T_j^i$  at every point in its proper frame and then try to transform the many proper frames to the Schwarzschild coordinate system. In practice such an infinity of measurements is impractical to say the least. It is more profitable to construct models with given  $T_j^i$  and compare the

predictions of such models with experiment.

To carry out a solution to Einstein's gravitational field equations, one assumes a spherically symmetrical mass density in Schwarzschild coordinates  $T_4^4$ . He then integrates equation 3.2 or 3.13 to find the function  $a$ . Substituting the value of

$$\begin{aligned} J &= 1 - \frac{a}{r} \\ &= 1 - \frac{ax}{r_o(1-x)} \end{aligned} \quad 3.15$$

and its derivatives into 3.7 or 3.14, one obtains a second order linear homogeneous differential equation which, with the aid of the boundary conditions 3.8b and 3.9b or 3.11, may be solved for  $y$ . The metric tensor is then given by

$$g_{ij} = \begin{vmatrix} -\frac{1}{c^2} \frac{1}{J} & 0 & 0 & 0 \\ 0 & -\frac{1}{c^2} r^2 & 0 & 0 \\ 0 & 0 & -\frac{1}{c^2} r^2 \sin^2 \theta & 0 \\ 0 & 0 & 0 & y^2 J \end{vmatrix} ; \quad 3.16$$

the pressure of the fluid is given by

$$\begin{aligned} p &= \frac{c^4}{8\pi G} \frac{2Jy'}{ry} - c^2 T_4^4 \\ &= -\frac{c^4}{8\pi G} \frac{2Jx^3}{r_o^2(1-x)y} y_x - c^2 T_4^4 ; \end{aligned} \quad 3.17$$

and, if one assumes that the fluid is an ideal gas, the temperature is given by

$$\begin{aligned}
 T &= \frac{\omega}{R} \frac{P}{T_4} = \frac{\omega c^2}{R} \left[ \frac{2rJy'}{ya'} - 1 \right] \\
 &= - \frac{\omega c^2}{R} \left[ \frac{c^2}{8\pi G} \frac{2Jx^3}{r_o^2(1-x)yT_4} y_x + 1 \right] ,
 \end{aligned}
 \tag{3.18}$$

where  $\omega$  is the molecular weight of the material in the medium and  $R$ , the universal gas constant, is  $8.31662 \times 10^3 \text{ J/(K mole-}^\circ\text{K)}$ . The simplest way to determine the critical values of the radius, density, or mass (those values at which the space-time becomes physically non-realizable) is to note when  $J$  and  $y$  become zero.

One could proceed to obtain the regular and null geodesic equations for the space, but if the mass density is large enough to cause an appreciable deviation from the results already obtained by Schwarzschild for a vacuum, then the friction encountered by a material particle and the index of refraction of the material constituting the spherically symmetrical mass would prevent particles and light rays from following the geodesic curves.



## CHAPTER IV

### CALCULATIONS

#### A. Schwarzschild's Homogeneous Sphere

As our first example of the use of the previously derived equations for solving Einstein's gravitational field equation, let us consider a mass distribution having the following mass density:

$$T_4^4 = \begin{cases} 3 \frac{q}{r_o^2} \frac{c^2}{8\pi G} & \text{for } r \leq r_o \\ 0 & \text{for } r > r_o, \end{cases} \quad 4.1$$

where  $q$  is a dimensionless constant and  $r_o$  is a constant having the units of length. One sees immediately that the distribution represents a sphere of radius  $r_o$  composed of a medium having a constant density, surrounded by a complete vacuum. We shall assume that the pressure goes to zero at the surface of the sphere.

Substituting the mass density 4.1 into equation 3.2 and 3.15, one obtains

$$a(r) = \begin{cases} \frac{q}{r_o^2} r^3 & \text{for } r \leq r_o \\ q r_o & \text{for } r > r_o, \end{cases} \quad 4.2$$

and

$$J = \begin{cases} 1 - q \frac{r^2}{r_o^2} & \text{for } r \leq r_o \\ 1 - q \frac{r_o}{r} & \text{for } r > r_o, \end{cases} \quad 4.3$$

so that equation 3.7 yields

$$y'' = \begin{cases} \frac{3 - 2J}{rJ} y' & \text{for } r \leq r_o \\ \frac{2r - 5a}{2r(r-a)} y' & \text{for } r > r_o. \end{cases} \quad \begin{matrix} 4.4a \\ 4.4b \end{matrix}$$

In order to solve 4.4a, one considers  $y = y(J)$  so that

$$\frac{d^2 y}{dJ^2} = - \frac{3}{2} \frac{1}{J} \frac{dy}{dJ},$$

which has the solution

$$y = C_1 J^{1/2} + C_2, \quad 4.5a$$

where  $C_1$  and  $C_2$  are constants of integration.

The first integration of 4.4b yields

$$y' = C_3 \frac{r^{5/2}}{(r-a)^{3/2}},$$

and the second integration yields

$$\begin{aligned} y = & - 2C_3 \frac{r^{5/2}}{(r-a)^{1/2}} + 5C_3 \left( \frac{r}{2} + \frac{3a}{4} \right) (r^2 - ar)^{1/2} + \\ & + 15C_3 \frac{a^2}{8} \ln \left[ \sqrt{r(r-a)} + r - \frac{a}{2} \right] + C_4. \end{aligned} \quad 4.5b$$

The boundary conditions 3.8b require that

$$C_3 = 0, \quad 4.6a$$

and

$$C_4 = 1; \quad 4.6b$$

the stipulation that the pressure (equation 3.17) be zero at  $r = r_o$  requires that

$$C_1 = -3C_2 J^{1/2}(r_o) ; \quad 4.6c$$

and the condition that  $y$  be continuous at  $r = r_o$  requires that

$$C_2 = -\frac{1}{2} . \quad 4.6d$$

Hence our solution becomes

$$y = \begin{cases} \frac{3}{2} \left[ \frac{1-q}{1-qr^2/r_o^2} \right]^{1/2} - \frac{1}{2} & \text{for } r \leq r_o \\ 1 & \text{for } r > r_o \end{cases} \quad 4.7$$

The metric tensor is now seen to be

$$\begin{aligned} g_{11} &= \begin{cases} -\frac{1}{c^2} \left( 1 - q \frac{r^2}{r_o^2} \right)^{-1} & \text{for } r \leq r_o \\ -\frac{1}{c^2} \left( 1 - q \frac{r_o^2}{r} \right)^{-1} & \text{for } r > r_o \end{cases} \\ g_{22} &= -\frac{1}{c^2} r^2 \\ g_{33} &= -\frac{1}{c^2} r^2 \sin^2 \theta \\ g_{44} &= \begin{cases} \frac{1}{4} \left[ 3(1-q)^{1/2} - \left( 1 - q \frac{r^2}{r_o^2} \right)^{1/2} \right]^2 & \text{for } r \leq r_o \\ 1 - q \frac{r_o}{r} & \text{for } r > r_o \end{cases} \\ g_{ij} &= 0 \quad \text{for } i \neq j . \end{aligned} \quad 4.8$$

The pressure of the fluid is

$$p = 3 \frac{q}{r_o^2} \frac{c^4}{8\pi G} \frac{(1 - qr^2/r_o^2)^{1/2} - (1 - q)^{1/2}}{3(1 - q)^{1/2} - (1 - qr^2/r_o^2)^{1/2}}, \quad 4.9$$

and if the fluid is an ideal gas, the temperature is

$$T = \frac{\omega c^2}{R} \frac{(1 - qr^2/r_o^2)^{1/2} - (1 - q)^{1/2}}{3(1 - q)^{1/2} - (1 - qr^2/r_o^2)^{1/2}}. \quad 4.10$$

The total gravitational mass is

$$M = \frac{4}{3} \pi r_o^3 T_4^4 = \frac{c^2}{2G} q r_o. \quad 4.11$$

When Schwarzschild first solved the problem of the homogeneous sphere, he began by considering the case of a complete vacuum, starting with Einstein's gravitational field equation and a spherically symmetric metric tensor. By introducing the properties of the vacuum early in his development, he was able to reduce the complexity of Einstein's equation and obtain a solution. Again starting with Einstein's equation and again making the appropriate simplifications in the early stages of development, he proceeded to obtain a solution for a homogeneous medium. Combining these solutions and applying the boundary condition that the pressure be zero at the surface of the sphere, he obtained the metric tensor 4.8 and pressure 4.9 for a sphere of constant density and finite radius. With considerably less travail, we were able to arrive at the same quantities through the utilization of the equations in chapter III.

### B. Centrally Dense Mass Distribution

Before proceeding with a discussion of the various attributes of the sectionally continuous mass distribution of the Schwarzschild homogeneous sphere, let us obtain a solution to Einstein's gravitational field equation for a continuous, extended mass distribution. We shall consider a mass distribution having the following mass density:

$$\begin{aligned} T_4^4 &= \frac{3 q r_o^2}{(r_o + r)^4} \frac{c^2}{8\pi G} \\ &= \frac{3q}{r_o^2} x^4 \frac{c^2}{8\pi G}, \end{aligned} \tag{4.12}$$

where  $r_o$  is a constant having the dimensions of length and is called the radius of the distribution, and  $q$  is a dimensionless constant determining the central density of a distribution of a given radius. Because of the sharp rise in density at the center of symmetry, this distribution is only a rough approximation to that of a star (being about as far off in one direction as Schwarzschild's homogeneous sphere is in the other; cf. figure 4.3 and 4.4). Although the density has fallen to 1/16'th its central value at  $r = r_o$ , only 1/8'th of the total mass of the distribution is located interior to this radius. Nevertheless this mass distribution is not uninteresting, for its solution contains many of the characteristics associated with other extended mass distributions, and by comparing it with Schwarzschild's homogeneous sphere, one may obtain considerable insight into the properties of more realistic mass distributions.

An integration of 3.13 gives

$$a(x) = q r_0 (1-x)^3, \quad 4.13$$

so that

$$J(x) = 1 - qx(1-x)^2, \quad 4.14$$

and equation 3.14 becomes

$$U y_{xx} + V y_x + W y = 0,$$

where

$$\begin{aligned} U &= 2x(1-x)[1 - qx(1-x)^2] \\ &= x[2 - 2(1+q)x + 6qx^2 - 6qx^3 + 2qx^4] \\ &= \sum_{i=0}^4 u_i x^{i+1}, \end{aligned} \quad 4.15$$

$$\begin{aligned} V &= -3qx(1-x)^2(1-3x) + 2(3-2x)[1 - qx(1-x)^2] \\ &= 6 - (4+9q)x + 31qx^2 - 35qx^3 + 13qx^4 \\ &= \sum_{i=0}^4 v_i x^i, \end{aligned}$$

$$\begin{aligned} W &= 2qx(1-x)(2-3x) - 2q(1-x)^2(1-3x) + 2q(1-x) \\ &= 12qx - 24qx^2 + 12qx^3 \\ &= \sum_{i=0}^4 w_i x^{i-1}, \end{aligned}$$

and where the constants  $u_i$ ,  $v_i$ ,  $w_i$  are obtained from the table 4.1.

From linear differential equation theory<sup>7</sup> it is known that if  $J \neq 0$ , then there exist two nontrivial linearly independent power series solutions of equation 4.15 of the form

$i$	$u_i$	$v_i$	$w_i$
0	2	6	0
1	$-(1+q)2$	$-(4+9q)$	0
2	$6q$	$31q$	$12q$
3	$-6q$	$-35q$	$-24q$
4	$2q$	$13q$	$12q$

TABLE 4.1

COEFFICIENTS OF SERIES EXPANSION

$$y = \sum_{n=0}^{\infty} c_n (x-x_0)^n \quad 4.16$$

and that these power series converge in some interval  $0 < |x-x_0| < R$  (where  $R > 0$ ) about  $x_0$ , if  $0 < x_0 < 1$ , and there exists at least one nontrivial solution of the form

$$y = \sum_{n=m}^{\infty} c_n (x-x_0)^n \quad 4.17$$

which converges in some interval  $0 < |x-x_0| < R$  about  $x_0$  if  $x_0 = 0$  or  $x_0 = 1$ . Furthermore, if there exists only one linearly independent solution of the form 4.17, then the other linearly independent solution is of the form

$$y_2 = \sum_{n=m}^{\infty} c_n^0 (x-x_0)^n + c^0 y_1(x) \ln |x-x_0|, \quad 4.18$$

where  $c_m^0 \neq 0$ ,  $c^0$  is a constant, and  $y_1(x)$  is the first solution of the

form 4.17.

Expanding about the point  $x_0 = 0$ , one finds that equation 4.15 takes on the form

$$\sum_{n=m}^{\infty} \sum_{i=0}^4 \left[ n(n-1)u_i + nv_i + w_i \right] c_n x^{n+i-1} = 0. \quad 4.19$$

If we define  $c_{m-s} = 0$  for all  $s < 0$ , then

$$\sum_{n=m}^{\infty} \sum_{i=0}^4 \left[ (n-i)(n-i-1)u_i + (n-i)v_i + w_i \right] c_{n-i} x^{n-1} = 0. \quad 4.20$$

Setting the coefficients of the powers of  $x$  equal to zero gives for all  $n \geq m$

$$\sum_{i=0}^4 \left[ (n-i)(n-i-1)u_i + (n-i)v_i + w_i \right] c_{n-i} = 0. \quad 4.21$$

For  $n = m$ ,

$$\left[ m(m-1)u_0 + mv_0 \right] c_m = 0, \quad 4.22$$

which for  $c_m \neq 0$  gives

$$m = 0$$

or 
$$m = 1 - \frac{v_0}{u_0} = -2.$$

Both of these possible values of  $m$  lead to the same solution, namely

$$y_1 = \frac{1}{c_0} \sum_{n=0}^{\infty} c_n x^n. \quad 4.23$$

Therefore, the other linearly independent solution is of the form 4.18.

Hence the general solution is



$$y = C_1 y_1 + C_2 y_2 , \quad 4.24$$

where  $C_1$  and  $C_2$  are arbitrary constants. Our boundary condition 3.11 requires that

$$C_1 = 1 \quad 4.25$$

and  $C_2 = 0 .$

Since  $C_2 = 0$ , there is no need to evaluate the constants in equation 4.18 to obtain the second linearly independent solution of the differential equation.

To evaluate the constants in equation 4.23, one need only solve equation 4.21 for  $c_n$ , remembering that  $c_0 = 1$  (effectively) and  $c_{-s} = 0$  for all  $s < 0$ , and that  $u_i, v_i, w_i$  are given by table 4.1. One finds

$$\begin{aligned} c_0 &= 1 \\ c_1 &= 0 \\ c_2 &= -\frac{3}{4} q \\ c_3 &= \frac{1}{60}(26 - 33q) q \\ c_4 &= \frac{(8 + 13q)(26 - 33q)}{960} q + \frac{129}{96} q^2 - \frac{1}{4} q \\ c_n &= \frac{1}{n(n-1)u_0 + v_0} \sum_{i=1}^4 \left[ (n-i)(n-i-1)u_i + (n-i)v_i + w_i \right] c_{n-i} . \end{aligned} \quad 4.26$$

Placing these values back into equation 4.23, one obtains the function  $y$ .

In order to display on only two graphs the characteristics of  $y$  and its derivative with respect to  $x$ ,  $y_x$ , for both the strong and weak field cases, as well as to present a compact form for obtaining these quantities

for values of  $q$  from  $1 \times 10^{-7}$  to its maximum possible value 4.44, the negative of the common logarithm of  $(1-y)$  was plotted against  $x$  in figure 4.1, and the negative of the common logarithm of  $(-y_x)$  was plotted against  $x$  in figure 4.2.

The data for these figures were obtained from the I B M 1620 computer model 2 using the programs reproduced in appendices 1 and 2. The program in appendix 1 computes the coefficients 4.26, multiplies them by the appropriate power of  $x$ , and performs the summations

$$y = \sum_{n=0}^N c_n x^n \quad 4.27$$

and

$$y_x = \sum_{n=0}^N n c_n x^{n-1}.$$

For the weak field case ( $q < 0.1$ ), these series converge, and eight place accuracy is obtained by setting  $N$  equal to 20, a process requiring less than three minutes of computer time.

Because of the limited range of convergence of equation 4.23 for large values of  $q$ , it was necessary, in the strong field case, to evaluate  $y$  and  $y_x$  by another method. One could have performed a new series expansion of the form 4.16 near the limit of convergence of 4.23, using the values obtained in the older expansion to evaluate the arbitrary constants occurring in the new series. However, because of the accessibility of the computer and the lack of necessity for great accuracy, it was deemed more expedient to have the computer solve equation 4.15 directly.

$-\log(1-y)$

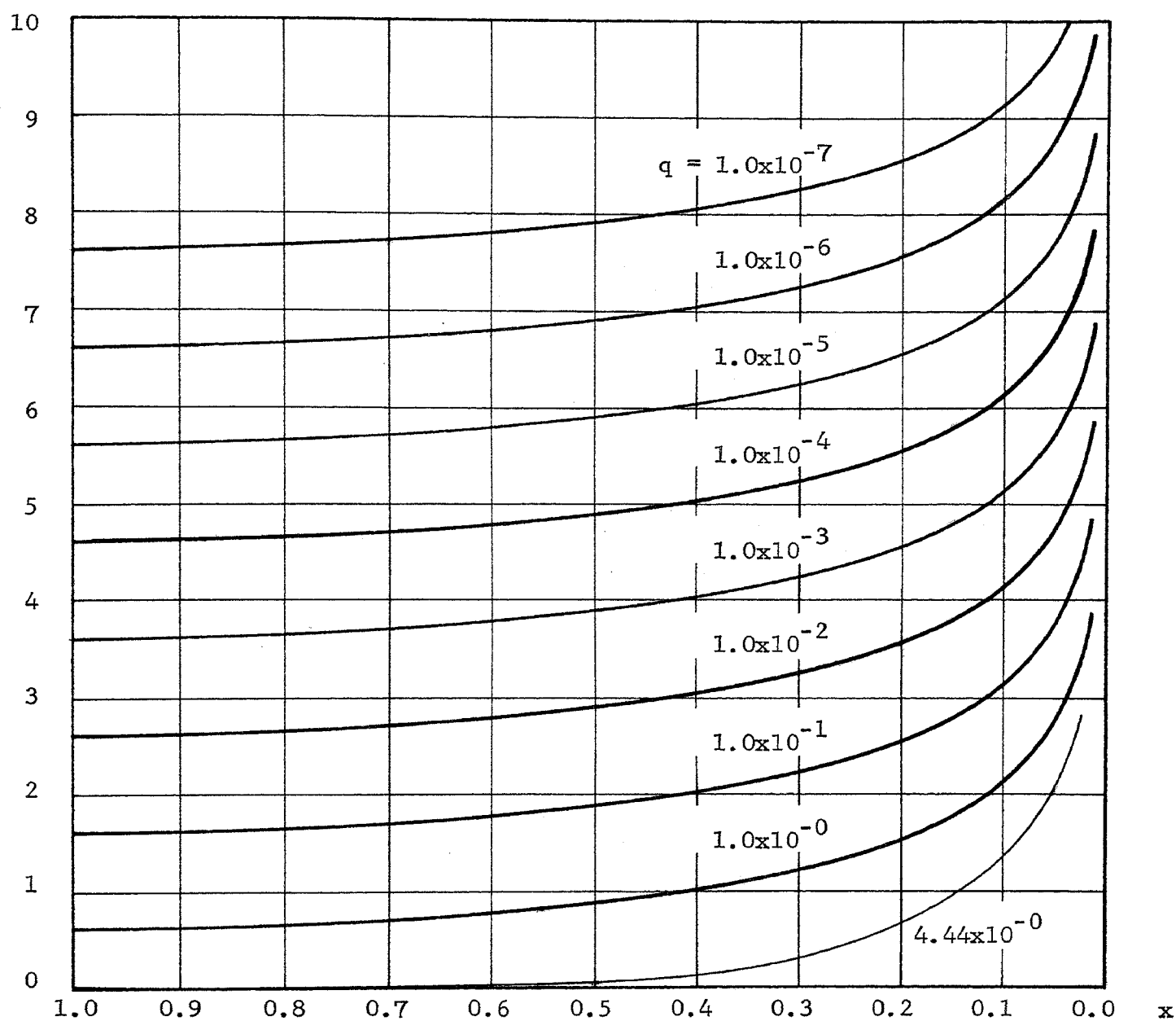


FIGURE 4.1

CURVE OF  $-\log(1-y)$  VERSUS  $x$

$-\log(-y_x)$

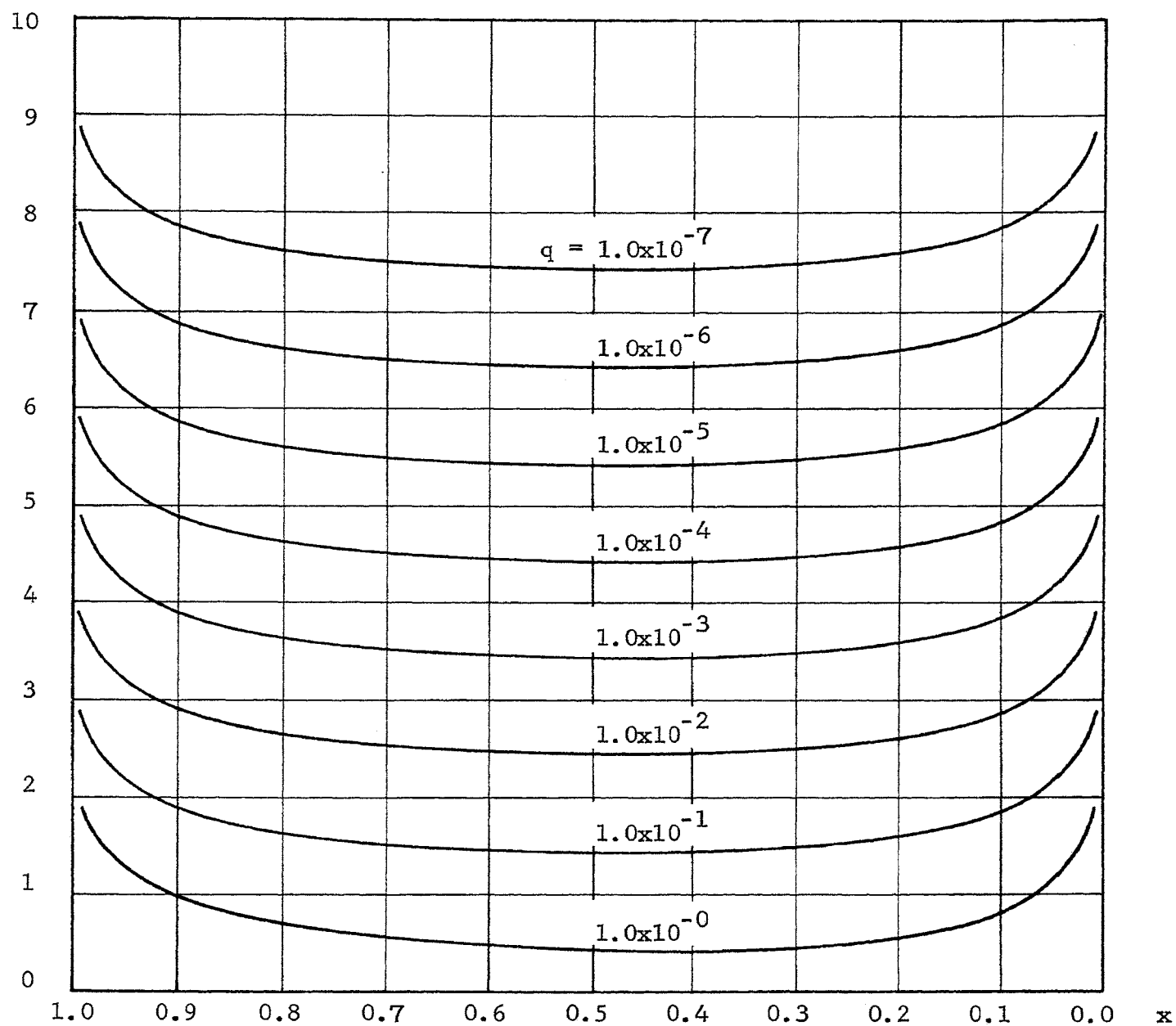


FIGURE 4.2

CURVE OF  $-\log(-y_x)$  VERSUS  $x$

Letting  $y$ ,  $y_x$ ,  $y_{xx}$ ,  $y_{xxx}$  be the values of the respective functions for the current value of  $x$ , then approximately

$$\begin{aligned}
 y(x+dx) &= y + y_x dx + \frac{1}{2} y_{xx} dx^2 + \frac{1}{6} y_{xxx} dx^3 \\
 y_x(x+dx) &= y_x + y_{xx} dx + \frac{1}{2} y_{xxx} dx^2 \\
 y_{xx}(x+dx) &= -\frac{1}{U} (V y_x + W y) \\
 y_{xxx}(x+dx) &= -\frac{1}{U} \left[ (U_x + V) y_{xx} + (V_x + W) y_x + W_x y \right] ,
 \end{aligned}
 \tag{4.28}$$

(the last equation being obtained through the differentiation of equation 4.15) where  $dx$  is a small increment in the independent variable. Through a method of iteration one is able to obtain the function  $y$  and its derivative  $y_x$  for integral multiples of  $dx$ , starting with the initial values

$$\left. \frac{d^m y}{dx^m} \right|_{x=0} = \frac{1}{m!} c_m ,
 \tag{4.29}$$

where the  $c_m$  are given by equation 4.26. Appendix 2 is a reproduction of the program used to perform this iteration. With 1000 iterations, one is able to obtain four to five place accuracy using fifteen minutes of computer time.

The metric tensor for the centrally dense mass distribution is given by equation 3.16, the pressure by

$$p = \frac{c^4}{8\pi G} \frac{3q}{r_o^2} \left[ \frac{2}{3q} \frac{Jx^3}{(1-x)y} (-y_x) - x^4 \right] ,
 \tag{4.30}$$

the temperature by

$$T = \frac{\omega c^2}{R} \left[ \frac{2}{3q} \frac{J}{x(1-x)y} (-y_x) - 1 \right] ,
 \tag{4.31}$$

and the total mass by

$$M = \frac{c^2}{2G} q r_o . \quad 4.32$$

### C. Comparison of the Solutions

When one compares the homogeneous sphere and the centrally dense mass with the mass distributions found in nature, he finds that neither distribution occurs naturally. The homogeneous sphere, being an "all or nothing" distribution, contains no provisions for continuity, a characteristic which nature deems almost necessary, and implies too great a mass inside the radius  $r = r_o$  and too little outside; whereas the centrally dense mass, although continuous, possesses an unnatural "lump" at its center of symmetry, and implies too great a total mass outside the radius  $r = r_o$  and too little inside. The two solutions, therefore, are somewhat complementary, each possessing qualities the other lacks.

Before comparing our solutions with nature, let us list the values of certain properties associated with some stars, the best known examples of naturally occurring spherically symmetric mass distributions large enough to possess an appreciable gravitational field. Although there are many different kinds of stars, the vast majority belong to a classification called the "main sequence", the rest being classified as "giants", "super giants", or "white dwarfs". Table 4.2 gives the

Name	Type	$r_o/r_{os}$	$M/M_s$	$q$
Beta Centauri	Main Sequence	11.0	25.	$2.5 \times 10^{-5}$
Sun	Main Sequence	1.0	1.0	$1.1 \times 10^{-5}$
Barnard's Star	Main Sequence	0.16	0.18	$1.2 \times 10^{-5}$
van Biesbroeck's Star	Main Sequence	0.01	0.1	$1.1 \times 10^{-4}$
Capella	Giant	12.	4.2	$3.9 \times 10^{-6}$
Beta Pegasi	Giant	170.	9.	$5.7 \times 10^{-7}$
Antares	Super Giant	480.	30.	$6.7 \times 10^{-7}$
Sirius B	White Dwarf	0.034	0.96	$3.9 \times 10^{-4}$
van Maanen's Star	White Dwarf	0.007	0.14	$2.2 \times 10^{-4}$

TABLE 4.2

## PROPERTIES OF THE STARS

names of several typical stars, their classifications, their relative radii, their relative masses, and their  $q$ -values (cf. page 42).<sup>9</sup>

Our Sun, a typical star occurring in the "middle" of the main sequence, may be used to illustrate properties characteristic of most of the other stars in our galaxy. Table 4.3 lists the accepted<sup>8</sup> values for several constants associated with the Sun.

Let us now compare the density of our Sun with that of our two solutions to Einstein's equation. Figures 4.3 and 4.4 are plots of the relative density  $T_4^4/\rho_c$  as a function of  $r$  and  $x$ , respectively, where  $\rho_c$

quantity	symbol	value
total mass	$M_s$	$1.9866 \times 10^{30} \text{ kg}$
radius	$r_s$	$6.957 \times 10^8 \text{ m}$
mean molecular weight	$\omega_s$	0.590
central density	$\rho_{cs}$	$90.0 \text{ gm/cm}^3$
central temperature	$T_{cs}$	$13.7 \times 10^6 \text{ }^\circ\text{K}$
central pressure	$p_{cs}$	$1.74 \times 10^{16} \text{ n/m}^2$

TABLE 4.3

## CONSTANTS ASSOCIATED WITH THE SUN

is the density at the center of symmetry. Schwarzschild's homogeneous sphere is represented by the heavy curve, the centrally dense mass by the medium curve, and the Sun by the thin curve.<sup>8</sup> The Sun has a constant mass density near its center which drops rapidly near the radius  $r_0$  and which, although very small, extends on out to infinity approaching zero asymptotically.

Since neither of the calculated distributions approximates that of a star very closely, it is difficult to evaluate the two arbitrary constants  $q$  and  $r_0$  in terms of the constants normally associated with the star. When constructing theoretical models, one, of course, wants to choose the constants so as to obtain the "best fit" with nature, i.e. so that the calculations based on the models match as closely as possible



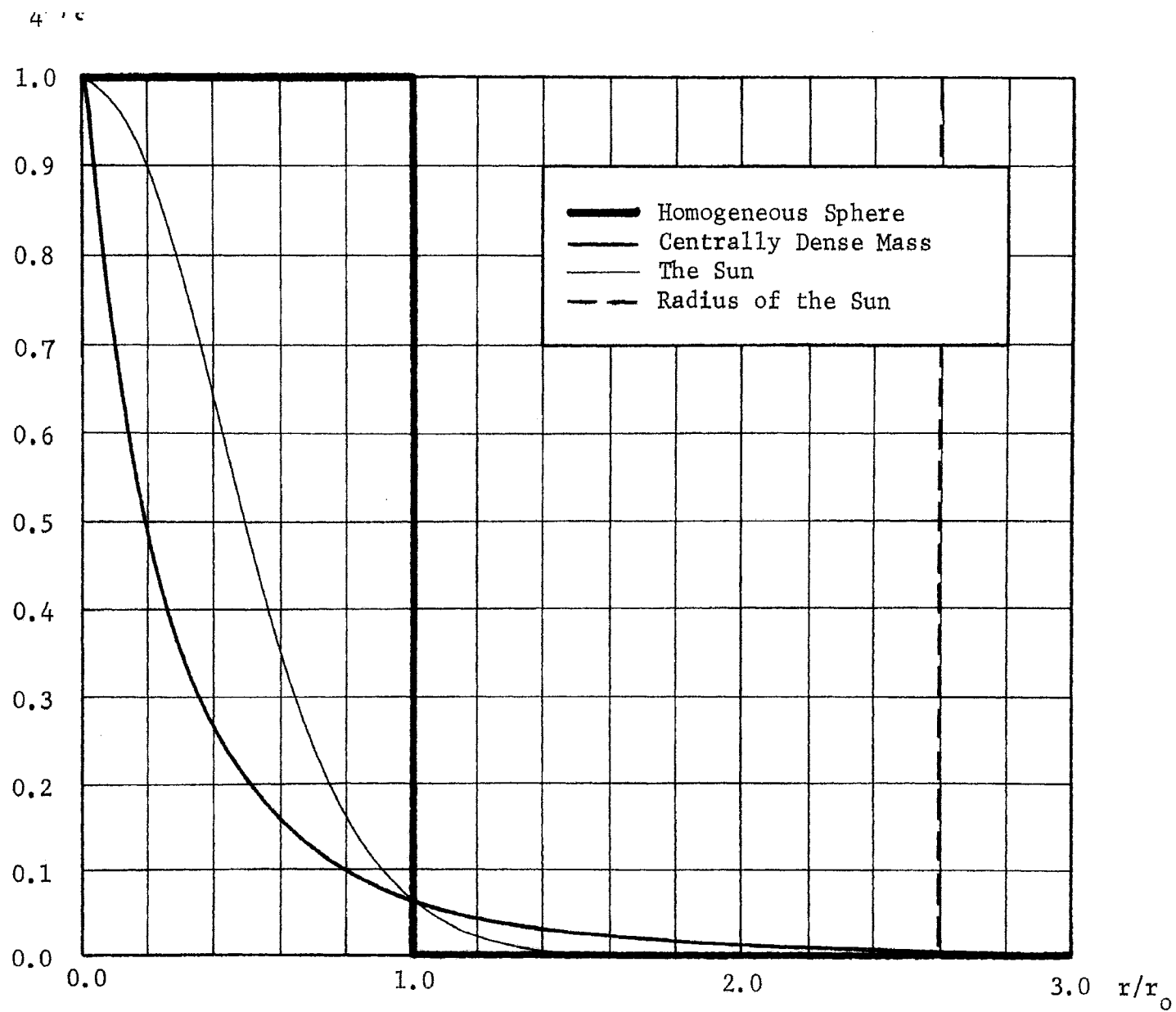


FIGURE 4.3

CURVE OF RELATIVE DENSITY VERSUS RADIUS

$$T_4^4/\rho$$

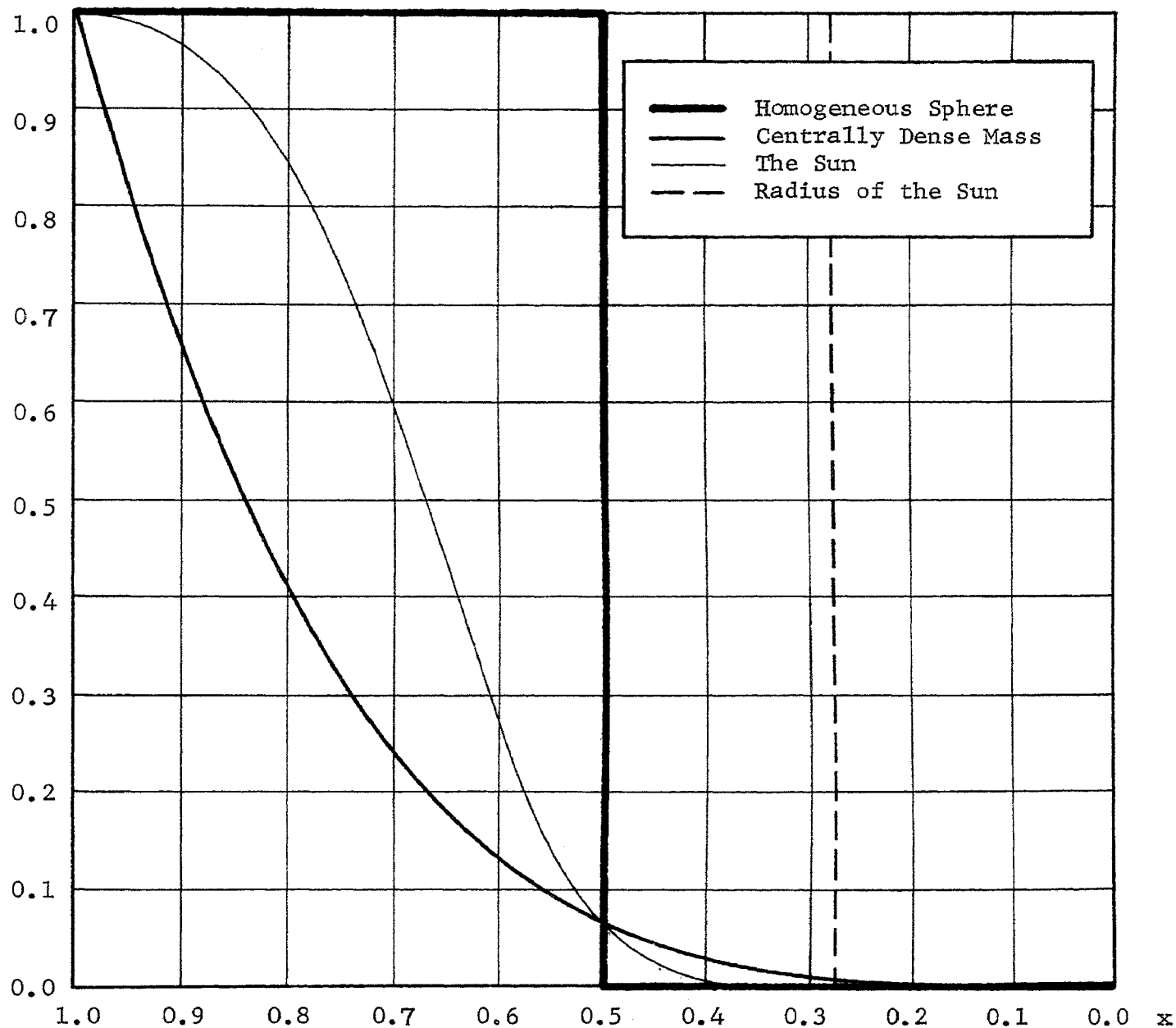


FIGURE 4.4

CURVE OF RELATIVE DENSITY VERSUS  $x$

the observed phenomena.

Rather than define two different  $r_o$ 's, so as to obtain a "best fit" for each of the models, we shall define the  $r_o$  occurring in both models to be that radius at which the density has fallen to one-sixteenth its value at the center of symmetry. By choosing  $r_o$  in this manner, at least for our Sun and probably for all stars, the relative density of the star will lie everywhere between the relative densities of the two models. For the Sun, this radius is:

$$r_{os} = 0.385 r_s = 2.68 \times 10^8 \text{ meters} . \quad 4.33$$

Since one would expect the relative densities of a star to be essentially the same as that of the Sun, we shall assume  $r_o$  is always 0.385 times the radius of the star; since the experimentally measured mass of a star is its total gravitational mass, we shall equate the constant  $M$  to the known mass of the star; and since the value of  $q$  in both models is given by

$$q = \frac{2G M}{c^2 r_o} , \quad 4.34$$

we shall make this a definition of the "q-value" for any star.

When one computes the central density  $\rho_c$ , he finds that both models give

$$\rho_c = \frac{M}{\frac{4}{3}\pi r_o^3} . \quad 4.35$$

In the case of the Sun, the central density would be:

$$\rho_{cs} = 24.5 \text{ gm/cm}^3, \quad 4.36$$

which is considerably below the value shown in table 4.3. If one had chosen  $r_{os} = 0.25 r_s$ , however, the central densities predicted by the models would have agreed with the value in the table. This choice would also have brought the density of the homogeneous sphere into closer agreement with that of the Sun, but would have caused the densities of the centrally dense mass and the Sun to become more separated.

Although the density of the Sun lies everywhere between those of our models, its gravitational mass does not. Figure 4.5 is a graph of the gravitational mass divided by the total mass plotted as a function of  $x$  for the homogeneous sphere (heavy line), the centrally dense mass (medium line), and the Sun (thin line).<sup>8</sup> Since the mass is proportional to the function  $a$  (cf. section III.A.), the figure is also a plot of this function (in units of  $2GM/c^2$ ). Surprisingly, one notices that the homogeneous sphere and the Sun are in relatively good agreement. The centrally dense mass, however, has entirely too much mass located at large distances from the center of symmetry.

It was mentioned earlier that the general theory of relativity predicts that certain mass distributions exhibit singular regions as one moves toward or away from the center of symmetry and that for almost all spherically symmetric mass distributions there exists a maximum physically allowable "size". The two functions determining whether or not a particular distribution is physically realizable are the functions

$M(x)/M$

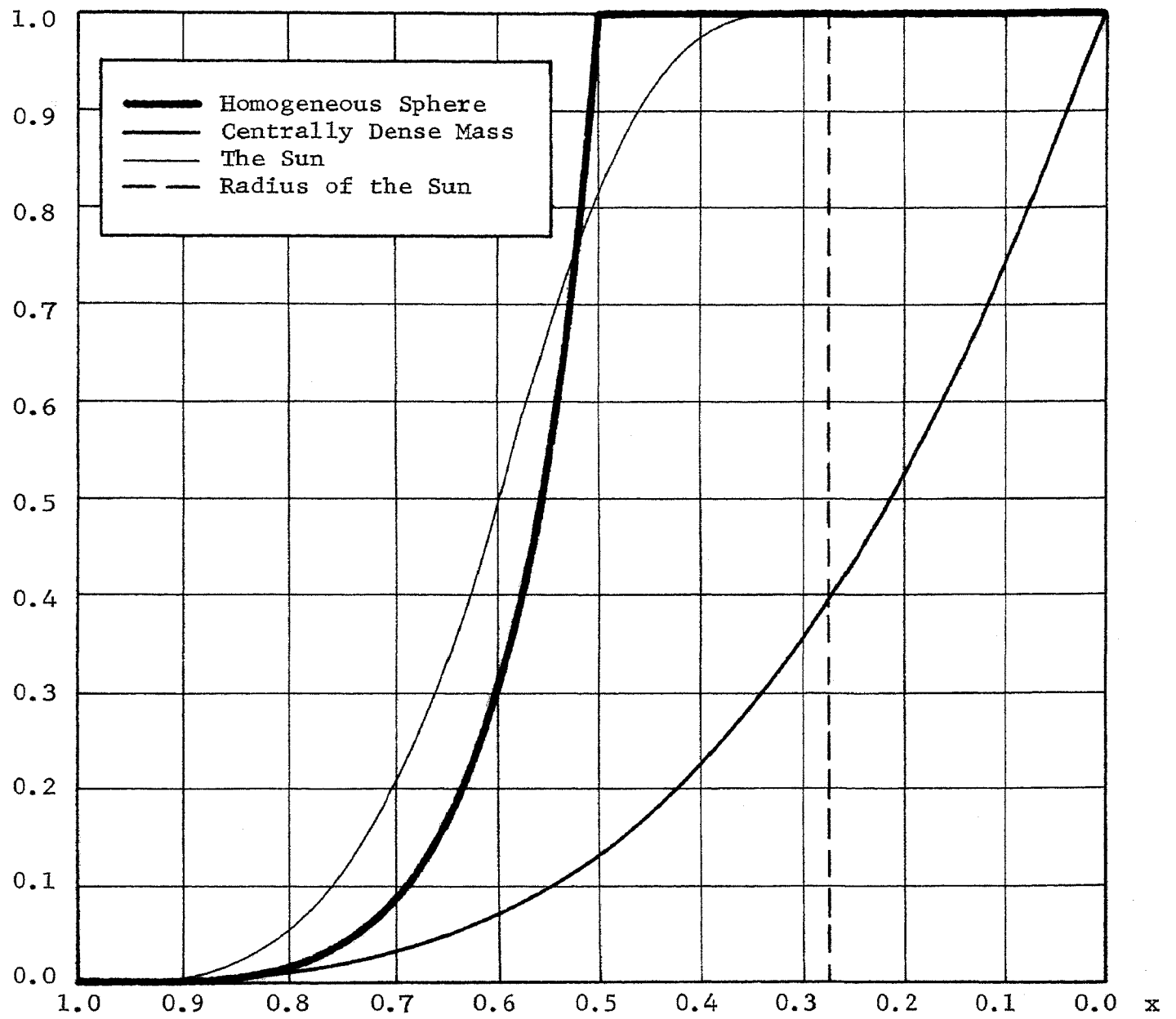


FIGURE 4.5

CURVE OF MASS VERSUS  $x$

y and J, for if either one becomes zero at any point, the space-time is not of uniform character. We, therefore, need to determine the conditions under which

$$\begin{aligned} y &= 0, \\ \text{and } J &= 0. \end{aligned} \tag{4.37}$$

Figure 4.6 shows J as a function of x for the particular case  $q = 0.75$ . The curve is a typical example of a physically realizable spherically symmetrical mass distribution. At the center of symmetry ( $x = 1$ ), J is essentially constant and equal to unity. As one moves away from the center of symmetry, J decreases to a minimum and then rises again to approach unity asymptotically. As the constant q is increased, the minimum value of J decreases until finally for some value  $q_J$  the minimum becomes zero.

For the homogeneous sphere the minimum value of J occurs at  $x = \frac{1}{2}$ , or  $r = r_0$ , namely at the surface of the sphere, and the value of q is

$$q_J = 1; \tag{4.38a}$$

for the centrally dense mass, the minimum value of J occurs at  $x = \frac{1}{3}$ , or  $r = 2r_0$ , and the value of q is

$$q_J = \frac{27}{4} = 6.75. \tag{4.38b}$$

The y-function of the two models have been plotted in figure 4.7 for  $q = 0.75$ . It is interesting to note that both solutions have zero slopes at both  $x = 0$  and  $x = 1$  (true for all values of q), and

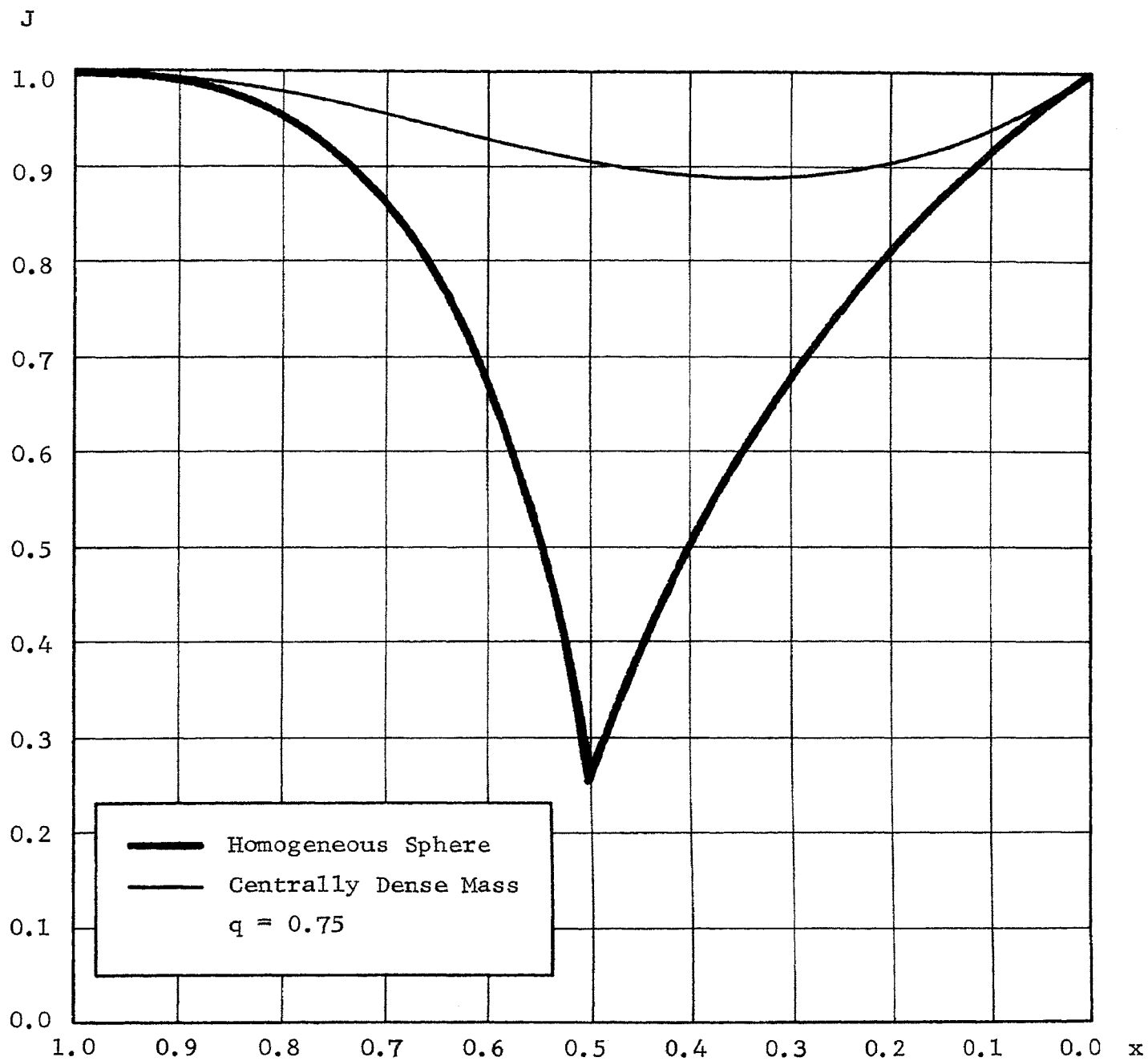


FIGURE 4.6

CURVE OF  $J$  VERSUS  $x$

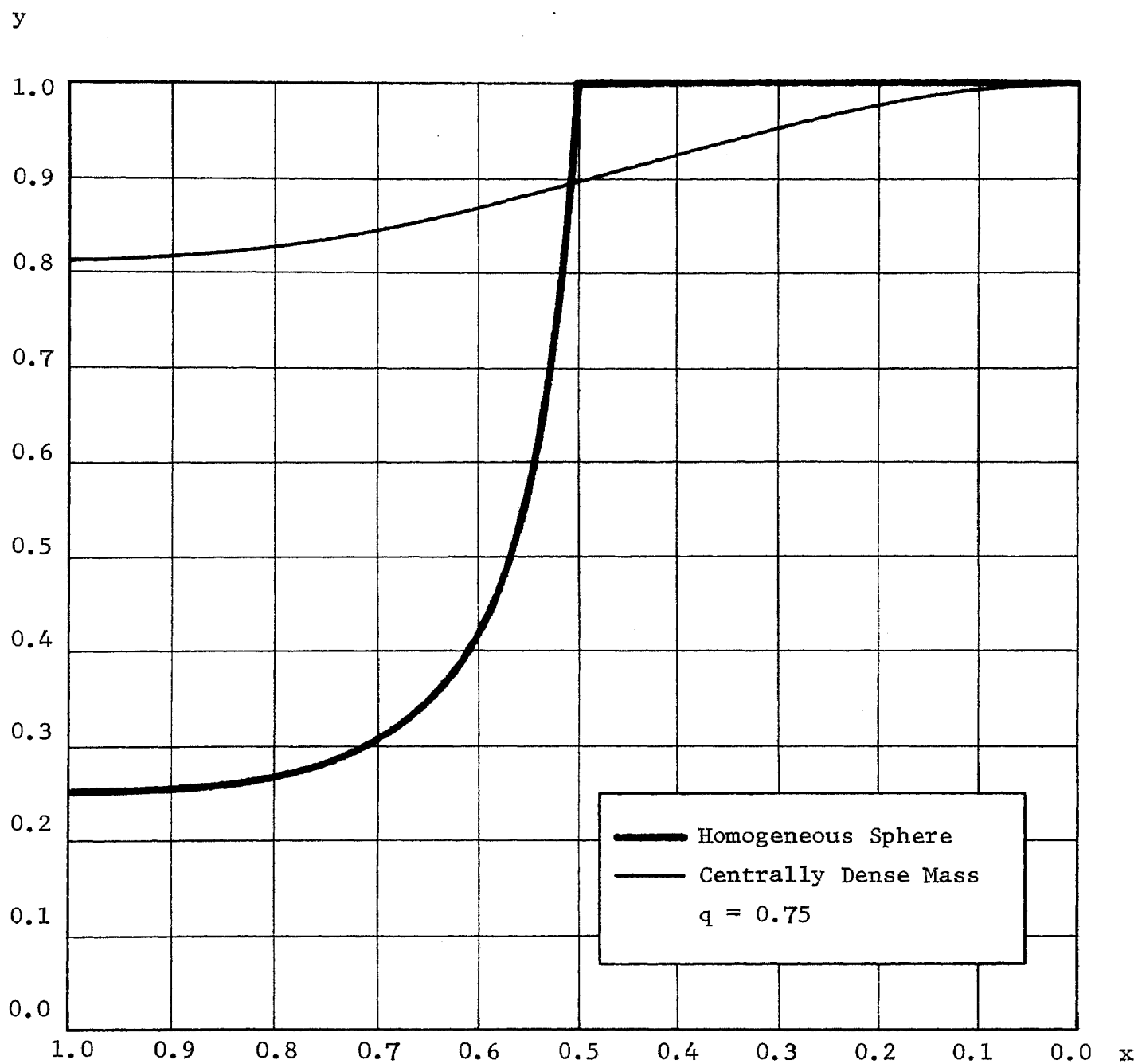


FIGURE 4.7

CURVE OF y VERSUS x



that  $y \leq 1$  for all values of  $x$  having a minimum value at  $x = 1$ . As the constant  $q$  is increased, the minimum value of  $y$  decreases until finally for some value  $q_y$  it becomes zero.

For the homogeneous sphere this value of  $q$  is

$$q_y = \frac{8}{9} ; \quad 4.39a$$

and for the centrally dense mass the value of  $q$  is

$$q_y = 4.44 . \quad 4.39b$$

Since neither  $y$  nor  $J$  may be zero, the smallest of the two values  $q_J$  and  $q_y$  is the true "critical value" of  $q$  for which the space-time becomes non-uniform. One notices that in both models the critical value  $q_c$  occurs when  $y$  first becomes zero ( $J$  still being positive):

$$q_c = q_y = 0.889 , \quad (\text{homogeneous sphere}) \quad 4.40a$$

$$q_c = q_y = 4.44 . \quad (\text{centrally dense mass}) \quad 4.40b$$

Since  $q_y$  occurs at the center of symmetry and results in only the  $g_{44}$  (time-like) component of the metric tensor becoming unrealistic, one suspects that the property,

$$q_c = q_y , \quad 4.41$$

is generally applicable to spherically symmetric static perfect fluids.

Because one expects a true star to have properties similar to the two cases being considered, one should expect to find no star for which the value of  $q$  exceeds the neighborhood of unity. Table 4.2 confirms

this expectation for the few stars listed, and there is no indication that mass distributions do exist for which  $q$  even approaches the critical value (with the possible exception of quasi-stellar sources for which the masses and radii are not well known).

It is interesting to note that neither the average density, the total mass, nor the radius of a star is sufficient in itself to determine whether or not a distribution is realizable, but the ratio of the mass to the radius is. If one increases the mass (holding the radius constant), decreases the radius (retaining the original mass), or increases the radius (maintaining a constant average density) he eventually reaches the critical stage. As a star cools off and contracts, its  $q$ -value increases; however, such dying stars appear to remain well below the critical stage.

Since the four-four component of the metric tensor may be used to relate the proper times of rest objects, it may be used to compute the relative rate of chemical reactions, or nuclear reactions, at various depths within the star. During a coordinate change  $dt$ , a proper time lapse of  $d\tau_\infty = dt$  will take place at large distances from the star while a proper time lapse of

$$d\tau_r = \sqrt{g_{44}} dt \quad 4.42$$

will take place at a distance  $r$  from the center of symmetry. As a result, if one second passes for someone a large distance from a star,  $\sqrt{g_{44}}$  seconds will pass inside the star. Similarly, if  $R_\infty$  is the rate of a nuclear reaction taking place a large distance from the star, then

the rate  $R$  of the reaction taking place a distance  $r$  from the center of the star is given by

$$R = R_{\infty} \sqrt{g_{44}} . \quad 4.43$$

Since  $\sqrt{g_{44}}$  is always less than 1, reactions will take place at a slower rate inside the star. (The gravitational red shift, although not a "nuclear" reaction, is essentially the same phenomenon.)

Figure 4.8 is a plot of the function  $\sqrt{g_{44}}$  as a function of  $x$ . One sees that there is considerably more time dilation in the case of the homogeneous sphere than for the centrally dense mass even though the masses and radii may be equal. This is true because the homogeneous sphere has most of its mass located in a small volume, thereby causing a large "warp" in the space. The centrally dense mass, on the other hand, has most of its mass dispersed, producing a lesser "warp".

Figure 4.9 shows the pressure of a star plotted against the variable  $x$  for  $q = 0.75$ . Although there are distinct similarities between these pressure curves and the density curves of figure 4.4, the differences, which determine the temperature of the distribution, are very important. In the homogeneous sphere, the pressure is seen to rise very rapidly just inside the surface but taper off to an essentially constant value near the center. This is only reasonable, for as one approaches the center of the sphere, the source of the pressure gradient (i.e. the gravitational attraction of all the mass interior to that radius) becomes weaker until at the center of symmetry

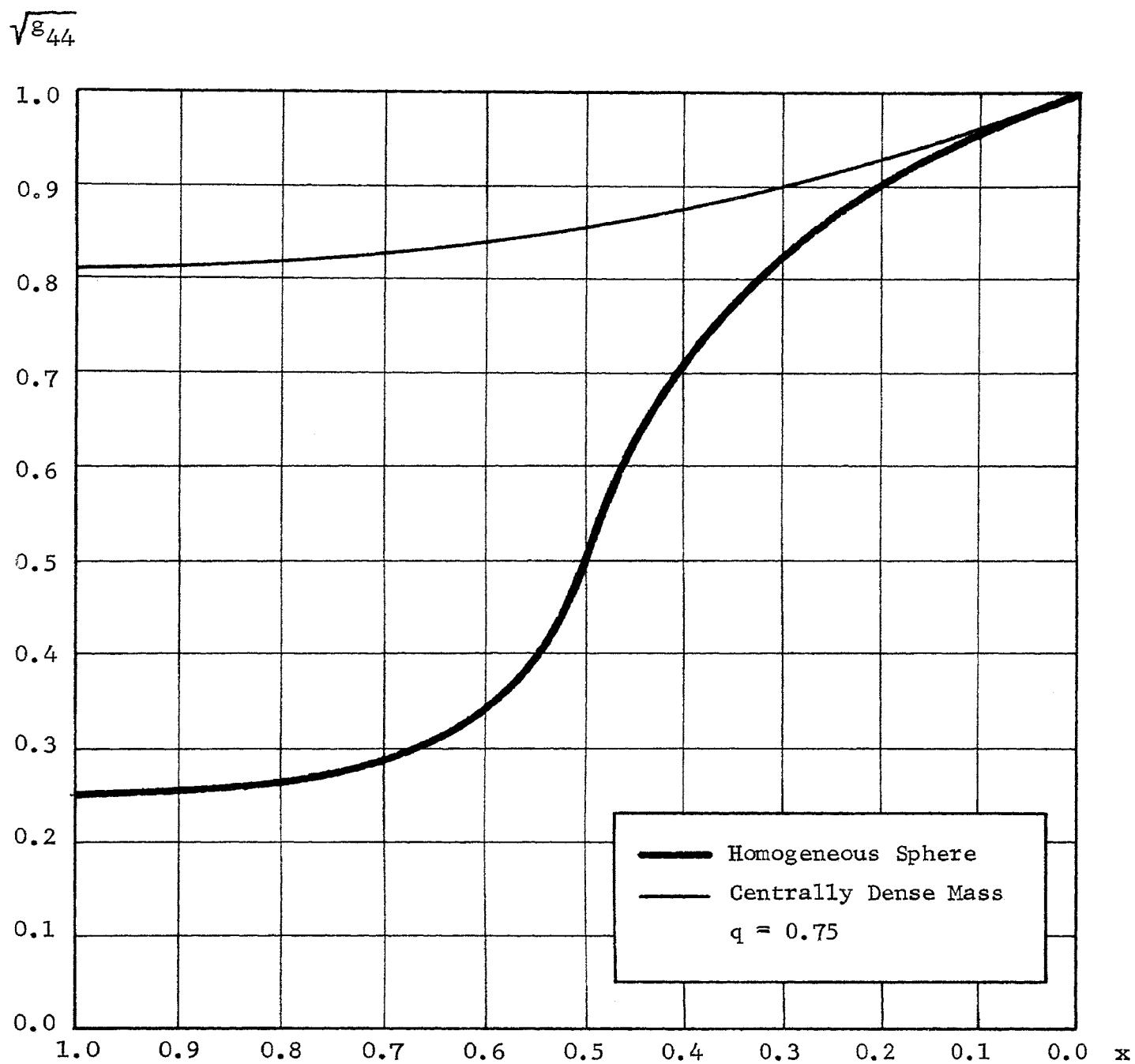


FIGURE 4.8

CURVE OF SQUARE ROOT OF  $g_{44}$  VERSUS  $x$

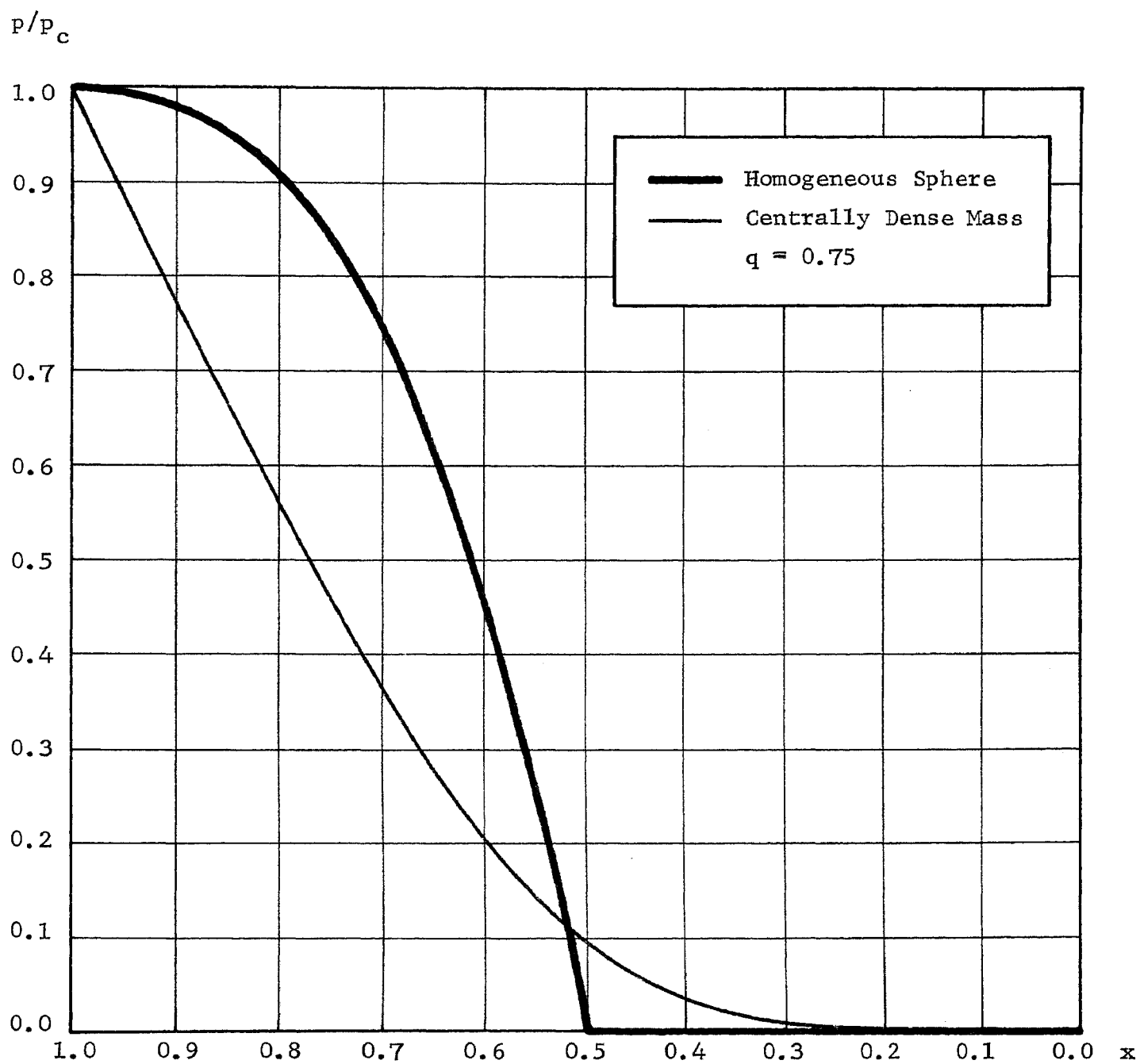


FIGURE 4.9

CURVE OF RELATIVE PRESSURE VERSUS  $x$

the pressure gradient becomes zero. In the case of the centrally dense mass, on the other hand, the density increases fast enough as one approaches the center of symmetry to overcome this tendency of the pressure gradient to become zero. Nevertheless, the pressure of the centrally dense mass does not rise as rapidly near the center of symmetry as does the density.

Because the homogeneous sphere is more compact than the centrally dense mass, its gravitational field is stronger and its pressure at the center of symmetry is greater. For our Sun, the central pressure is found to be

$$p_c = 1.39 \times 10^{16} \text{ n/m}^2 \quad (\text{homogeneous sphere}) \quad 4.44a$$

and 
$$p_c = 0.11 \times 10^{16} \text{ n/m}^2, \quad (\text{centrally dense mass}) \quad 4.44b$$

which are only slightly less than the accepted value given in table 4.3. Undoubtedly, even closer agreement would have been obtained in the case of the homogeneous sphere if we had chosen  $r_0$  to be 0.25 the radius of the sun instead of 0.385.

Figure 4.10a shows the relative temperature of a star plotted as a function of  $x$  for  $q = 0.75$ . The curve for the homogeneous sphere is the same as the corresponding pressure curve (figure 4.9) just as one would expect. The centrally dense mass, on the other hand, gives results that at first glance may seem surprising: The temperature, starting from absolute zero at a large distance from the center of symmetry, rises to a maximum near  $r = r_0$ , and then decreases as one continues toward

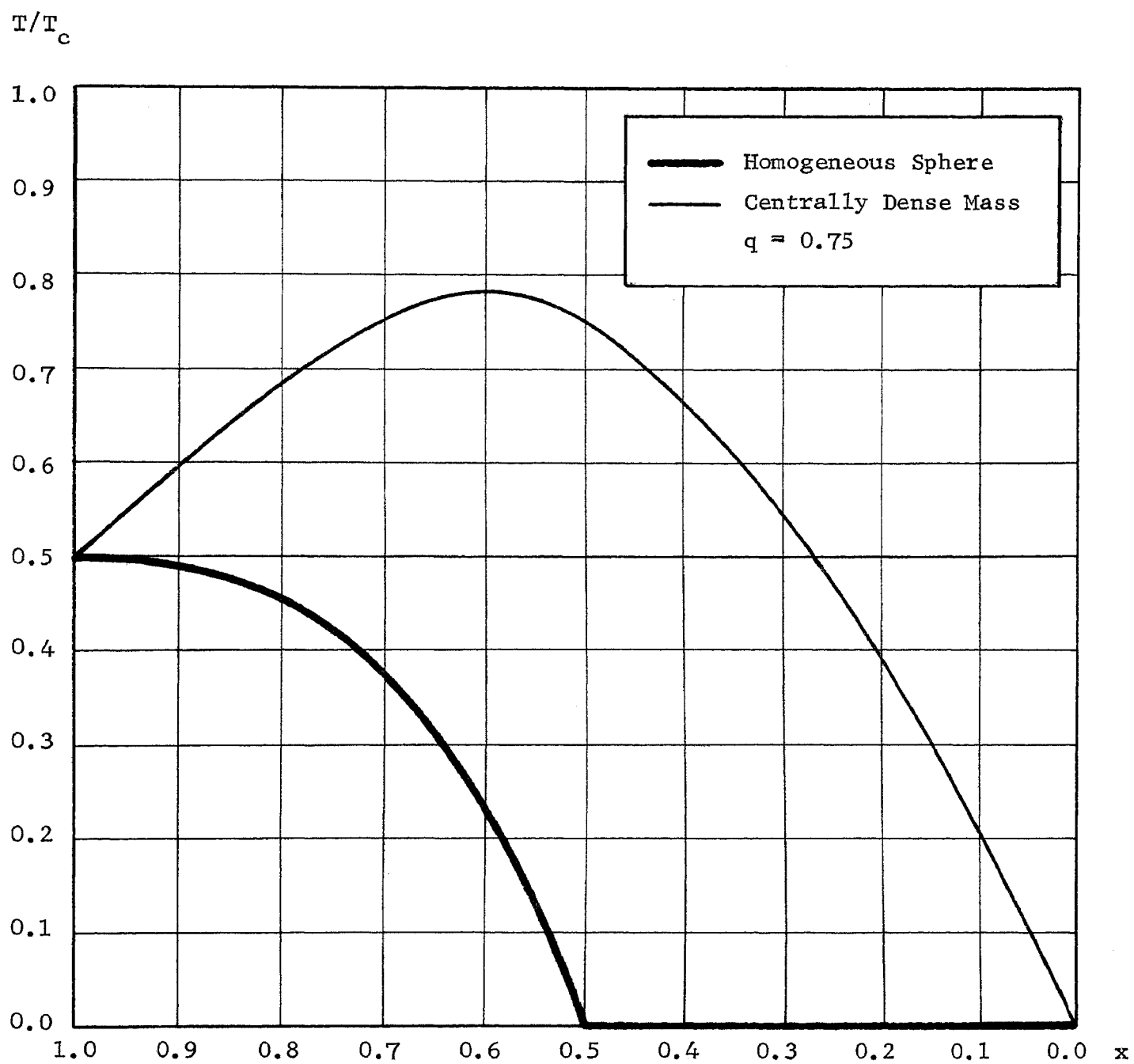


FIGURE 4.10a

CURVE OF RELATIVE TEMPERATURE VERSUS  $x$

the center of symmetry. The reason for this "cooling off" as one approaches the center of symmetry is connected with the fact that near the center of symmetry the pressure does not increase as rapidly as the density. Since the ideal gas law may be written as

$$\frac{p}{\rho} = \frac{R}{\omega} T , \quad 4.45$$

then one sees the temperature must decrease.

Figure 4.10b shows the absolute temperature plotted as a function of  $x$  for our two models of the Sun and for the Sun itself. It should be said that the curve for the centrally dense mass (medium thick curve) is based on inconclusive computations. Equation 4.31, from which the computations were made, requires the subtraction of unity from the quantity

$$\frac{2J}{3qx(1-x)y} (-y_x) ,$$

which differed from unity in about the eighth place. Since  $y_x$  and  $y$  were known only to eight significant figures (the limit set by the computer), the error in computing  $T$  was only slightly less than  $T$  itself. This difficulty was not encountered in the strong field case ( $q = 0.75$ ), because an increase in the  $q$ -value, corresponding to a contraction of the star, is accompanied by a rise in temperature. This rise in temperature as a star contracts is observed experimentally (at least for the surface of such stars).

When one compares the accepted value of the central temperature



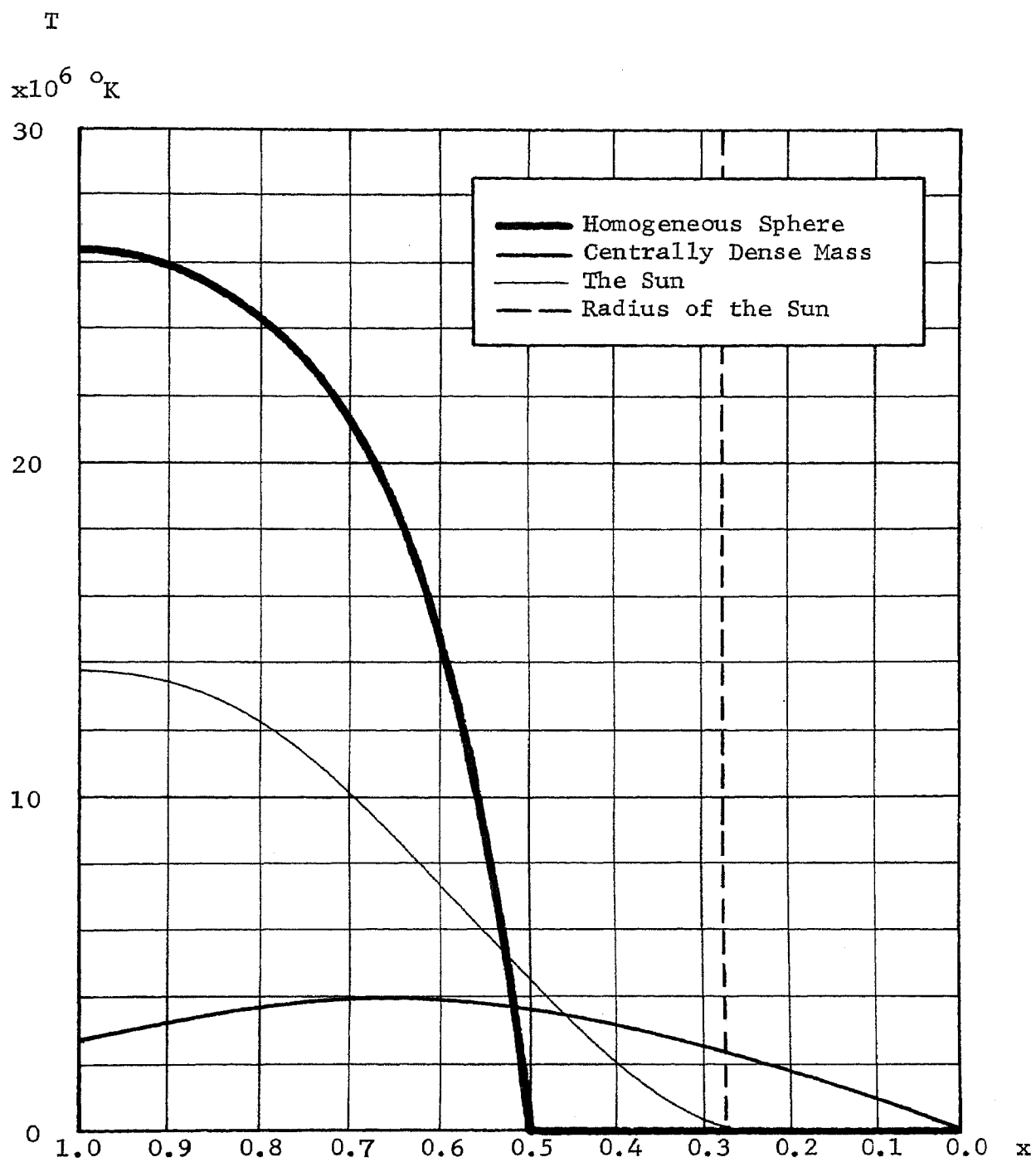


FIGURE 4.10b

CURVE OF ABSOLUTE TEMPERATURE VERSUS x

of the Sun found in table 4.3 with that computed from the two models,

$$T_c = 26.4 \times 10^6 \text{ } ^\circ\text{K} \quad (\text{homogeneous sphere}) \quad 4.46a$$

and  $T_c = 2.7 \times 10^6 \text{ } ^\circ\text{K} , \quad (\text{centrally dense mass}) \quad 4.46b$

he finds it lying between them. Indeed, except for a small region near  $r = r_0$ , the Sun's temperature lies everywhere between the temperatures computed from the two models.

This would seem to indicate that a model whose mass distribution more closely approximated that of the Sun would give results comparing favorably with the Sun.

#### D. Other Distributions

We saw in the previous section that results based on the two solutions that have been obtained for Einstein's field equation seem to be in qualitative agreement with physical observations even though neither of the models chosen was expected to represent a physically observed mass distribution. One wonders, therefore, how difficult it would be to solve Einstein's equation for a distribution whose mass density approximates that of a star more closely than our previous models. The main difficulty in choosing a single analytic expression for the mass density of the star, is that such an expression results in so complicated a J function that equation 3.14 becomes unwieldy. For example the relatively simple mass distribution

$$T_4^4 = 3q \frac{r_o^4}{(r_o^3 + r^3)^2} \frac{c^2}{8\pi G} \quad 4.47$$

results in a differential equation of the form 4.15 where U, V, W are polynomials of degree 12 instead of degree 4 as in the centrally dense mass distribution.

Analytically it is much simpler to break the mass distribution into two expressions, one valid inside the radius  $r_o$ , the other valid outside this radius. The internal density could be expressed as a constant minus a power of  $r$  or of  $(1-x)$ , i.e.

$$T_4^4 = \frac{c^2}{8\pi G} \frac{3q}{r_o^2} \left[ 1 - \left( \frac{r}{r_1} \right)^a \right] \quad 4.48a$$

or

$$T_4^4 = \frac{c^2}{8\pi G} \frac{3q}{r_o^2} \left[ 1 - \left( \frac{1-x}{1-x_1} \right)^a \right], \quad 4.48b$$

and the external density could be expressed as an inverse power of  $r$  or a power of  $x$ , i.e.

$$T_4^4 = \frac{c^2}{8\pi G} \frac{3q}{r_o^2} \left( \frac{r_2}{r} \right)^b \quad 4.48c$$

or

$$T_4^4 = \frac{c^2}{8\pi G} \frac{3q}{r_o^2} \left( \frac{x}{x_2} \right)^b, \quad 4.48d$$

where the constants  $r_1, x_1, r_2, x_2$  are chosen so that  $T_4^4$  and its derivative are continuous at  $r = r_o$ .

Neither 4.48a, 4.48b, nor 4.48c could represent mass distributions for an entire space, because 4.48a and 4.48b imply negative mass densities

when  $r > r_1$  and when  $x < x_1$ , and 4.48c implies an infinite mass density at the center of symmetry. Distributions of the form 4.48d are of the centrally dense type. (We have already considered the special case of  $b = 4$ .)

In order that there be a finite total gravitational mass, then in both equations 4.48c and 4.48d, the constant  $b$  must be greater than 3. This means that regardless of the mass distribution, at large distances from the center of symmetry the density must fall off faster than the inverse cube in order that the total mass be finite. (Our centrally dense mass just met this criterion and, as a result, had a considerable portion of its total mass located a great distance out.)

One might consider mass densities proportional to a positive power of the radius, but all such distributions have infinite total mass in addition to being physically unreasonable for large bodies.

Perhaps the most satisfactory approach to obtaining an accurate solution for a physically realizable distribution is to expand the internal density in a series of positive powers of  $r$  or  $(1-x)$ , and the external density in a series of negative powers of  $r$  or of positive powers of  $x$ . One should not have to take too many terms before obtaining a fairly good representation of the density, and the solution, although tedious, would be rather straightforward.

## CHAPTER V

### CONCLUSION

By reducing Einstein's gravitational field equations to a single second order linear homogeneous differential equation, we have simplified considerably the travail necessary to solve Einstein's equation for the case of a spherically symmetric static perfect fluid. The general procedure, outlined in section III.D., has been illustrated with the two examples: (1) the homogeneous sphere, a sectionally continuous mass distribution first solved by Schwarzschild, and (2) the centrally dense mass, a mass distribution described by a single analytic expression.

Qualitatively the two models predicted properties for the system much in accordance with our expectations. The pressure and temperature were everywhere positive or zero (the zero occurring at large distances from the center of symmetry); time was found to proceed at a slower rate in the stronger gravitational field (gravitational red shift and slow nuclear reaction rate); the critical mass-radius ratio (the  $q$ -value) was well above the observed values for the stars; the curves for such functions as  $a$ ,  $y$ ,  $J$  were continuous and, in the case of the centrally dense mass, had continuous derivatives.

Quantitatively the homogeneous sphere was the more accurate of the two models of the Sun. This is not surprising considering the fact that only one-eighth of the total mass of the centrally dense mass is

internal to  $r = r_o$ , whereas for the Sun one expects around eight-tenths of the total mass to be inside that radius.

All in all, the results of the computations based on the homogeneous sphere and the centrally dense mass are encouraging and indicate that a solution based on a more appropriate mass distribution would yield results comparing favorably with experiment.

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## APPENDIX

## APPENDIX 1

Program for Evaluation of the Series Expansion of

$y$  and  $y_x$

```
C      Y-FUNCTION FOR CENTRALLY DENSE STAR
      DIMENSION U(4),V(4),W(4),C(5),X(101),Y(101,3),Y1(101,3)
      READ 100,NDIV,MTERM,MORE
      PRINT 500,NDIV,MTERM,MORE
      X(1) = 0.0
      NMAX=MTERM+3
      N1 = NMAX + MORE
      N2 = NMAX + 1
      IMAX = NDIV+1
      DMAX = IMAX - 1
      DX = (1.0-X(1))/DMAX
10  Y(1,1) = 0.0
      Y1(1,1) = 0.0
      DO 5 I=2,IMAX
      X(I) = X(I-1) + DX
      Y1(I,1) = 0.0
5  Y(I,1) = 0.0
      READ 200,T
      U(1) = -2.0-2.0*T
      U(2) = 6.0*T
      U(3) = -6.0*T
      U(4) = 2.0*T
      V(1) = -4.0-9.0*T
      V(2) = 31.0*T
      V(3) = -35.0*T
      V(4) = 13.0*T
      W(1) = 0.0
      W(2) = 12.0*T
      W(3) = -24.0*T
      W(4) = 12.0*T
      C(1) = 0.0
      C(2) = 0.0
      C(3) = 0.0
      C(4) = 1.0
      N = 5
```

```

15 D = 0.0
   P = N
   DO 20 I=1,4
     S=I
     I2 = 5-I
20 D = ((P-S-4.0)*(P-S-5.0)*U(I)+(P-S-4.0)*V(I)+W(I))*C(I2)+D
   C(5) = -(D/((P-4.0)*(2.0*(P-5.0)+6.0)))
   K = N - 4
   K2 = N-5
   IF (N-NMAX) 24,24,27
24 IF (K-1) 21,21,22
21 DO 23 I=1,IMAX
   Y1(I,1) = Y1(I,1) + (P-4.)*C(5)
23 Y(I,1) = Y(I,1) + C(5)*X(I)
   GO TO 35
22 DO 25 I=1,IMAX
   Y1(I,1) = Y1(I,1) + (P-4.)*C(5)*X(I)**K2
25 Y(I,1) = Y(I,1) + C(5)*X(I)**K
   GO TO 35
27 IF (N-N2) 30,30,29
30 DO 31 I=1,IMAX
   Y1(I,2) = Y1(I,1)
31 Y(I,2) = Y(I,1)
29 DO 28 I=1,IMAX
   Y1(I,2) = Y1(I,2) + (P-4.)*C(5)*X(I)**K2
28 Y(I,2) = Y(I,2) + C(5)*X(I)**K
35 N = N+1
   C(1) = C(2)
   C(2) = C(3)
   C(3) = C(4)
   C(4) = C(5)
   IF (N-N1) 15,15,41
41 M1 = N-MORE-4
   M2 = M1 + MORE
   PRINT 300,T
   PRINT 301,M1,M2,M1,M2
   DO 45 I=1,IMAX
45 PRINT 400,X(I),Y(I,1),Y(I,2),Y1(I,1),Y1(I,2)
   GO TO 10
100 FORMAT (7I10)
200 FORMAT (4E18.8)
300 FORMAT (1H1//7X,3HQ =E15.8)
301 FORMAT (6X,1HX,11X,1HYI5,12X,1HYI5,11X,2HYI5,11X,2HYI5)
400 FORMAT (3X,F7.4,4E18.8)
500 FORMAT (5X,6HNDIV =I4,6X,7HMTerm =I6,6X,6HMORE =I4)
END
      20          10          10
      1.1E-05

```

## APPENDIX 2

Program for Evaluation of  $y$  and  $y_x$  through the Method of Iteration

```

C      Y-FUNCTION FOR CENTRALLY DENSE STAR      (DIRECT SOLUTION)
      READ 100,NDIV,NSKIP
1  READ 101,T
      XX = 0.0
      Y = 0.0
      Y1 = 0.0
      Y2 = -3.*T/2.
      Y3 = (26.-33.*T)*T/10.
      DMAX = NDIV
      DX = 1./DMAX
      DMAX = NSKIP
      DELX = DMAX*DX
      PRINT 200
      PRINT 201
      PRINT 202,T,NDIV,NSKIP
      PRINT 203,DX,DELX
      PRINT 300
      I = 1
      J = 0
10  IF (Y) 11,12,13
11  YABS = -Y
      GO TO 14
12  YABS = 0.00000001
      GO TO 14
13  YABS = Y
14  YLOG = 0.43429448*LOGF(YABS)
      IF (Y1) 16,17,18
16  Y1ABS = -Y1
      GO TO 19
17  Y1ABS = 0.00000001
      GO TO 19
18  Y1ABS = Y1
19  Y1LOG = 0.43429448*LOGF(Y1ABS)
      YY = 1.+Y
      PRINT 301,XX,YY,YLOG,Y1,Y1LOG
      J = J+1
20  I = I+1
      Z = XX
  
```

```

YL = Y
Y1L = Y1
Y2L = Y2
Y3L = Y3
U=2.*Z*(1.-Z)*(1.-T*Z*(1.-Z)**2)
V=6.-(4.+9.*T)*Z+31.*T*Z**2-35.*T*Z**3+13.*T*Z**4
W=12.*T*Z*(1.-Z)**2
U10=2.-4.*(1.+T)*Z+18.*T*Z**2-24.*T*Z**3+10.*T*Z**4
V10=-(4.+9.*T)+62.*T*Z-105.*T*Z**2+52.*T*Z**3
W10=12.*T-48.*T*Z+36.*T*Z**2
XX = Z+DX
Y = YL+Y1L*DX+Y2L/2.*DX**2+Y3L/6.*DX**3
Y1 = Y1L+Y2L*DX+Y3L/2.*DX**2
IF (U) 30,25,30
25 U=0.00000001
30 Y2 = -(V*Y1L+W*(YL+1.))/U
Y3 = -((U10+V)*Y2L+(V10+W)*Y1L+W10*(YL+1.))/U
IF (I-NDIV-1) 40,10,1
40 IF ((I-1)/(J*NSKIP)) 50,20,10
50 PRINT 500
100 FORMAT (2I10,E20.8)
101 FORMAT (4E18.8)
200 FORMAT (1H117X,35HY-FUNCTION FOR CENTRALLY DENSE STAR)
201 FORMAT (28X,15HDIRECT SOLUTION)
202 FORMAT (6X,3HT =E15.8,5X,6HNDIV =I6,5X,7HNSKIP =I6)
203 FORMAT (29X,4HDX =F8.4,5X,6HDELX =F7.4)
300 FORMAT (/4X,1HX11X,1HY14X,4HYLOG13X,2HY112X,5HY1LOG)
301 FORMAT (1X,F7.4,4E16.8)
500 FORMAT (6X,25HSOME BODY MADE A BLUE-PER)
END
1000      200
1.0

```

## VITA

The author was born in Cleburne, Texas, on July 20, 1938. He lived in and near Fort Worth, Texas, through the completion of his elementary education at R. L. Paschal High School in May 1956. Majoring in physics, he entered the Texas College of Arts and Industries, Kingsville, Texas, from which he graduated summa cum laude on May 28, 1960, obtaining the degree of Bachelor of Science. That September he entered the graduate school of the University of Missouri School of Mines and Metallurgy, studying under a three year fellowship granted through the National Defense Education Act, and later under a one year National Science Foundation fellowship.

Leaving Rolla in the summer of 1964, he spent the following nine months teaching physics at the Texas College of Arts and Industries.

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